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Investigation of stress distribution of changes of section of members under two dimensional bending and direct stress

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INVESTIGATION OF STRESS DISTRIBUTION AT CHANGES OF SECTION OF MEMBERS UNDER TWO DIMENSIONAL BENDING AND DIRECT STRESS.

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SUMMARY

The application of the complex variable to the theory of elasticity by Muschelišhvili and C.A.M. Gray, in cases where the shape being investigated can be transformed to the unit circle, leads to an infinite set of simultaneous equations in an infinite number of unknowns (x_n, say).

Results are normally only possible by successive approximation and then only if the equations are in a suitable form.

This thesis attempts to find solutions for various such sets, and particularly for the change of section shape.

In most cases, x_n does not approach 0 as n approaches infinity and so consideration of only a finite number of terms would not be expected to produce good results. However, by choosing suitable initial values for the x_n and inserting these in the equations, the infinite series so formed can often be determined analytically, and made to agree very closely with the right hand sides. In this way, solutions were found for various sets of equations.

The stresses in a long cantilever with concentrated end loads were completely determined. In the case of the change of section, the most difficult part, that of calculation of the sum of the principal stresses, has been

done, although not with such satisfactory accuracy as for the cantilever.

NOTATION

ψ = Airy stress function = $\frac{1}{8}(\bar{z}\Omega + z\bar{\Omega} + \omega + \bar{\omega})$

$\widehat{xx} = \sigma_x$ = normal stress in the x-dirn.

$\widehat{yy} = \sigma_y$ = normal stress in the y-dirn.

$\widehat{xy} = \tau_{xy}$ = shearing stress in y-dirn. on plane normal to x-dirn.

$\widehat{rr}, \widehat{\theta\theta}$, $\widehat{r\theta}$ equivalent stresses in polar coordinates

$\Omega = A_0 + A_1\sigma + A_2\sigma^2 + \ldots$

$\Omega' = b_0 + b_1\xi + b_2\xi^2 + \ldots$

$\omega' = B_0 + B_1\sigma + B_2\sigma^2 + \ldots = a_0 + a_1\xi + a_2\xi^2 + \ldots$

P = intensity of point force

z = complex variable in plane of shape transformed

ω = complex variable in plane of half-plane

ξ = complex variable in plane of unit circle

x_n = variables in simultaneous equations reduced to real eqns.

δx_n = amounts which have to be added to initial values x_n

P_n = right hand sides of first boundary equations

Q_n = right hand sides of second boundary equations

$u_n = \theta_n + i\phi_n$ = coefficient of ξ^n in z.

$b_n = \alpha_n + i\beta_n$ = coefficient of ξ^n in Ω

$y_n = x_n \pm \frac{4}{\pi}$ for cantilever

h = lesser width of change of section shape

k = greater width of change of section shape

$a = (\frac{k}{h})^2$

$A = \frac{k}{\pi}$

$A_{2n-1} = \theta_n$

$A_{2n} = \phi_n$

$\alpha = -\frac{1 - ai}{(a - i)}$

$B = \frac{2Ai(1-i)^{1/2}}{(a-i)^{1/2}}$

B_{nm} = coefficients of sets of eqns.

ΔR_N = amount which has to be added to l.h.s. of eqn. N (assuming initial values of the x_N) to make it equal the r.h.s.

λ_n = ... see p. 105

$A = 1 - \lambda_2 + \lambda_4 - \ldots$

$B = \lambda_1 - \lambda_3 + \lambda_5 - \ldots$

σ = value on closed contour in complex plane (usually unit circle)

θ = variable angle in polar coordinates

r = variable radius in polar coordinates

a = radius of circle

α = particular angle in polar coordinates

γ = particular angle in polar coordinates

γ = a contour of integration

X_ν, Y_ν = boundary point loads applied externally in x and y directions respectively

$F_1 + iF_2 = 4i\oint(X_\nu + iY_\nu)\,ds$

A = constant in Schwarz - Christoffel transformation

CONTENTS

VOLUME 1

SECTION 1

INTRODUCTION

Derivation of stress distribution in two dimensional elasticity can be made generally and analytically for simple cases only. In more complex cases, approximate and highly particularized solutions are often all that are available. In the case when the two-dimensional shape is a polygon, the Schwarz-Christoffel transformation enables us, in theory, to always derive a set of infinite simultaneous equations in an infinite number of unknowns (the mechanical computations may be difficult), the solutions of which completely determine the stresses.

Section 2 states some basic elasticity theory, without proof, and gives two simple examples which clarify that theory. Section 3 gives some more-advanced and lesser-known theory which is directly applicable to the method being employed. Section 4 solves 3 sets of equations which are similar to, but simpler than, those derived later. Section 5 applies the more sophisicated method of solution to a case previously solved, that of a beam under simple loading, and compares the results. Section 6 completely determines the stresses in an infinitely long cantilever with a concentrated load in the centre of the end. Section 7 then finds solutions of the set of equations for a beam with a discontinuous change of section, under two loading cases. The sum of the principal stresses can then be calculated at any point, though a great deal of mechanical computation may be involved.

The conclusion discusses the usefulness of this approach for finding solutions, and the problems involved.

SECTION 2

A BRIEF SUMMARY OF BASIC ELASTICITY THEORY

The solution of two-dimensional problems, without body forces, is reduced to the integration of the differential equation $\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0$ (biharmonic equation) having regard to the boundary conditions.

Then
$$\widehat{xx}=\frac{\partial^2\psi}{\partial y^2}$$
$$\widehat{yy}=\frac{\partial^2\psi}{\partial x^2}$$
$$\widehat{xy}=-\frac{\partial^2\psi}{\partial x\partial y}$$

(See ref.1 pp. 26,29 e.g.)

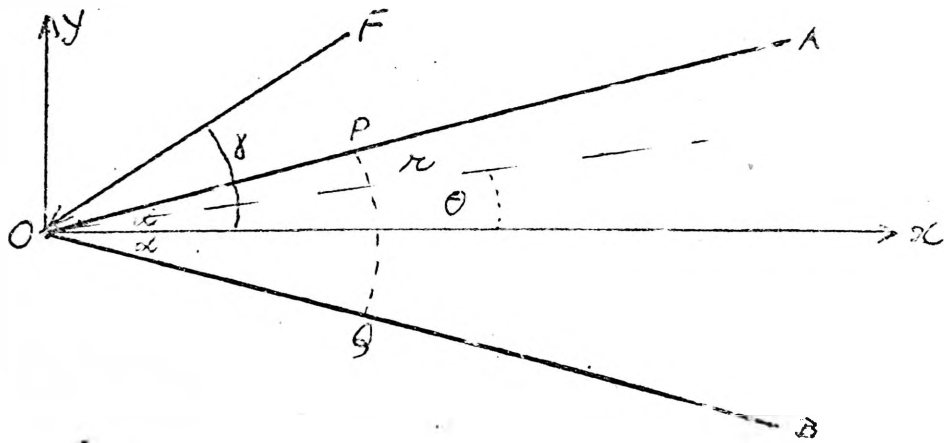
In polar coordinates, the above equations are

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)=0$$

and
$$\widehat{rr}=\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$
$$\widehat{\theta\theta}=\frac{\partial^2\psi}{\partial r^2}$$
$$\widehat{r\theta}=\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)$$

(See ref.1 pp. 55-57)

As an example, consider the case of a force at the apex of a wedge and then the particular case of a force on a half-plane.



Boundary conditions

when $\theta=\alpha$ $\qquad \widehat{\theta\theta}=\widehat{r\theta}=0$

when $\theta=-\alpha$ $\qquad \widehat{\theta\theta}=\widehat{r\theta}=0$

A stress function ψ which satisfies these boundary conditions and also satisfies the biharmonic is

$$\psi=r\theta(A\cos\theta-B\sin\theta)$$

Then $\widehat{\theta\theta}=0$ everywhere

$\widehat{r\theta}=0$ everywhere

$$\widehat{rr}=-\frac{2A\sin\theta}{r}-\frac{2B\cos\theta}{r}$$

(As $r\to 0$, $rr\to\infty$, so there must be a concentrated load at the apex)

If we take an arbitrary force F at angle γ

$$X=F\cos\gamma \qquad Y=F\sin\gamma \qquad \ldots (1)$$

Component in x-dirn. of force on PQ

$$X=\int_{-\alpha}^{\alpha}\widehat{rr}\,ds\cos\theta$$

simly. $Y=\int_{-\alpha}^{\alpha}\widehat{rr}\,ds\sin\theta$

so $X=-\int_{-\alpha}^{\alpha}\frac{2A\sin\theta+2B\cos\theta}{r_0}(r_0d\theta)\cos\theta$

(Note that on any arc PQ, resultant force is independent of r)

$$X=-B\int_{-\alpha}^{\alpha}2\cos^2\theta d\theta$$

$$=-B\left(+2\alpha+\frac{2\sin 2\alpha}{2}\right)$$

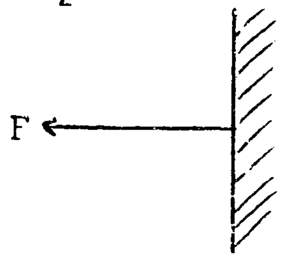
$$\left.\begin{aligned} &\underline{\underline{X=-B(2\alpha+\sin 2\alpha)}} \\ \text{simly. } &\underline{\underline{Y=A(\sin 2\alpha-2\alpha)}} \end{aligned}\right\} \quad \ldots (2)$$

Equating eqns.(1) and (2) gives

$$A=\frac{F\sin\gamma}{\sin 2\alpha-2\alpha} \qquad B=-\frac{F\cos\gamma}{2\alpha+\sin 2\alpha}$$

In particular, for a perpendicular force on the half-plane,

$\alpha = \frac{\pi}{2}$ and $\gamma = 0$



Then $A = 0$

$B = -\frac{F}{\pi}$

so $\psi = \frac{Fr\theta\sin\theta}{\pi}$

and $\widehat{rr} = \frac{2F}{\pi}\,\frac{1}{r}\cos\theta$

The complex variable has been applied to the mathematical theory of elasticity in a similar way, but more recently and to a lesser degree, as to the solution of Laplace's equation in two-dimensional cases of heat transfer, electric fields, etc..

Following Muschelišhvili (ref. 2),

let $\psi = \frac{1}{8}\left(\bar{z}\Omega + z\bar{\Omega} + \omega + \bar{\omega}\right)$

$$\Omega = f_1(z) \qquad \omega = f_2(z)$$

$$\nabla_1{}^2\psi = 4\frac{\partial^2\psi}{\partial z\,\partial\bar{z}} = \frac{1}{2}(\Omega' + \bar{\Omega}')$$

$=$ Real part of $\Omega'(z)$ and,

since the real or imaginary part of any function of a complex variable is a harmonic function, $\nabla_1{}^4\psi = 0$

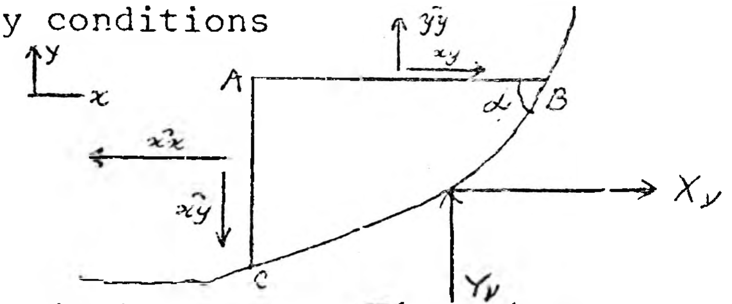
(so ψ satisfies the biharmonic).

Also $\nabla_1{}^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \widehat{yy} + \widehat{xx}$

similarly $\quad \widehat{xx} - \widehat{yy} - 2i\widehat{xy} = -\frac{1}{2}(\bar{z}\Omega'' + \omega'')$

So, knowing Ω' and ω'', we can find $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$.

Boundary conditions



$$4i\int(X_\nu + iY_\nu)ds = \Omega + z\overline{\Omega'} + \overline{\omega'} \quad \text{(ref. 2)}$$

The great advantage in using complex variables comes from the ease of using conformal transformations by simple substitution of new variables. The theory becomes even more advantageous when use is made of Cauchy's theorem to evaluate a regular function at any point within a closed contour as a contour integral around that contour.

The theorem states :- (ref. 3 p. 93)

If f(z) is regular within and on a closed contour C, and if ξ be a point within C, then

$$f(\xi) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-\xi}$$

Now if $\Omega(\sigma) = A_0 + A_1\sigma + A_2\sigma^2 + \ldots$ on the boundary C

$$\frac{1}{2\pi i}\oint \frac{\Omega(z)d\sigma}{\sigma-\xi} = \frac{1}{2\pi i}\int_C (A_0 + A_1\sigma + A_2\sigma^2 + \ldots)\frac{d\sigma}{\sigma-\xi}$$

$$= A_0 + A_1\xi + A_2\xi^2 + \ldots$$

$$= \Omega(\xi)$$

So if Ω is known on the boundary σ, we can find it at any internal point.

Also, if $\omega'_\sigma = B_0 + B_1\sigma + B_2\sigma^2 + \ldots$

and the boundary is a circle, centred at the origin, and $\sigma\bar{\sigma}=a^2$ i.e. $\sigma=ae^{i\theta}$ where a is the radius of the circle

$$\bar{\omega}' = \bar{B}_0 + \frac{\bar{B}_1 a^2}{z} + \frac{\bar{B}_2 a^4}{z^2} + \dots$$

Now $\frac{1}{2\pi i}\int \frac{1}{z}\,\frac{1}{z-\xi}\,dz = \frac{1}{2\pi i}\int \left(\frac{1}{z-\xi}-\frac{1}{z}\right)\frac{dz}{\xi}$

$$= \frac{1}{2\pi i}\left(\int_c \frac{1}{\xi}\,\frac{dz}{z-\xi} - \int_c \frac{dz}{\xi . z}\right)$$

$$= \frac{1}{\xi}-\frac{1}{\xi} = 0$$

Similarly $\frac{1}{2\pi i}\int \frac{1}{z^n}\,\frac{1}{z-\xi}dz = 0$ $\quad n=2,3,\dots$

so $\frac{1}{2\pi i}\int \bar{\omega}'\frac{dz}{z-\xi} = \frac{\bar{B}_0}{2\pi i}\int\frac{dz}{z-\xi}$

$$= \bar{B}_0$$

Now the boundary condition is

$$\Omega + z\bar{\Omega}' + \bar{\omega}' = 4i\int (X_\nu + iY_\nu)ds$$

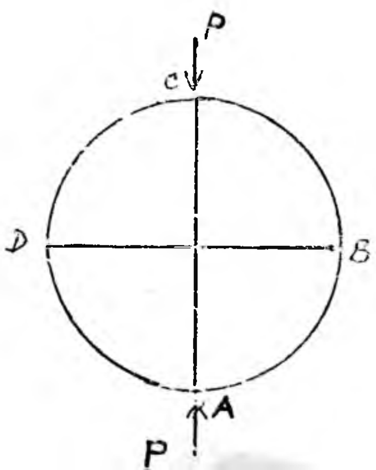
$= F_1 + iF_2$ (say) where F_1 and F_2 are known for any particular case. **(3)**

Taking $\frac{1}{2\pi i}\int \frac{dz}{z-\xi}$ of both sides yields

$$\Omega(\xi) + \frac{1}{2\pi i}\int_c Z\bar{\Omega}'\frac{dz}{z-\xi} + \bar{B}_0 = \frac{1}{2\pi i}\int_c (F_1+iF_2)\frac{dz}{z-\xi} \qquad (4)$$

As an example of the previous theory, consider the case of the unit circle with point loads at $\xi=i$, $\xi=-i$

$(Z=\xi)$ $\qquad$ ξ plane



The value of F_1+iF_2 can be taken as $-\frac{iP}{2}$ on CDA and $+\frac{iP}{2}$ on ABC

then $F_1+iF_2=4i\int(X_\nu+iY_\nu)ds=\begin{cases}2P \text{ on CDA}\\ -2P \text{ on ABC}\end{cases}$

$$\frac{1}{2\pi i}\int_c(F_1+iF_2)\frac{d\sigma}{\sigma-\xi} = \frac{1}{2\pi i}\left(\int_{\sigma_A}^{\sigma_C}-\frac{2Pd\sigma}{\sigma-\xi}+\int_{\sigma_C}^{\sigma_A}\frac{2Pd\sigma}{\sigma-\xi}\right)$$

$$=\frac{2iP}{\pi}\log\frac{\sigma_A-\xi}{\sigma_C-\xi}$$ (also see p. 16)

$$\Omega_z=A_0+A_1z+A_2z^2+\ldots$$

so $\Omega_z'=A_1+2A_2z+3A_3z^2+\ldots$

$$\overline{\Omega}_z'=\overline{A_1}+\frac{2\overline{A}_2}{z}+\frac{3\overline{A}_3}{z^2}+\ldots$$ noting $\bar{z}=1/z$ for the unit circle.

$$z\overline{\Omega}_z'=z\overline{A_1}+2\overline{A_2}+\frac{3\overline{A}_3}{z}+\ldots$$

and $\frac{1}{2\pi i}\int z\overline{\Omega}_z'\,\frac{dz}{z-\xi} = \overline{A_1}\xi+2\overline{A_2}$

so eqn. (1) becomes

$$\Omega(\xi)+\overline{A}_1\xi+2\overline{A}_2+\overline{B}_0 = -\frac{2iP}{\pi}\log\frac{\sigma_A-\xi}{\sigma_C-\xi}$$

differentiate with respect to ξ

$$\Omega'(\xi)+\overline{A}_1 = -\frac{2iP}{\pi}\left(-\frac{1}{\sigma_A-\xi}+\frac{1}{\sigma_C-\xi}\right)$$

$$=-\frac{2iP}{\pi}\left(\frac{1}{i+\xi}+\frac{1}{i-\xi}\right)$$

$$\Omega'(\xi)+\overline{A_1} = -\frac{4P}{\pi}\,\frac{1}{1+\xi^2} \qquad (5)$$

also $\Omega''(\xi) = \frac{8P}{\pi}\,\frac{\xi}{(1+\xi^2)^2}$

To find ω, we take the conjugate of eqn. (3).

VIZ $\quad \overline{\Omega}+\bar{z}\Omega'+\omega' = F_1-iF_2$

This gives, similarly to the above,

$$\overline{A}_0+2A_2+3A_3\xi+4A_4\xi^2+\ldots$$

$$+\omega'(\xi) = -\frac{2iP}{\pi}\log\frac{\sigma_A-\xi}{\sigma_C-\xi}$$

$$\omega''(\xi) = -\frac{4P}{\pi}\,\frac{1}{1+\xi^2}-(3A_3-4.2A_4\xi-5.3A_5\xi^2-\ldots)$$

Now $\Omega'(\xi) = A_1+2A_2\xi+3A_3\xi^2+4A_4\xi^3+\ldots$ by definition (6)

and $= -\bar{A}_1 - \frac{4P}{\pi}(1-\xi^2+\xi^4-\xi^6 \quad . \quad . \quad .)$ from equation (5) above

so $A_3 = \frac{1}{3}.\frac{4P}{\pi} \qquad A_4 = 0$

$A_5 = -\frac{1}{5}.\frac{4P}{\pi} \qquad A_6 = 0$

$A_7 = \frac{1}{7}.\frac{4P}{\pi} \qquad A_8 = 0$

so $\omega''(\xi) = -\frac{4P}{\pi}\,\frac{1}{1+\xi^2} - \frac{4P}{\pi}(1-3\xi^2+5\xi^4- \quad . \quad . \quad .)$

$= -\frac{4P}{\pi}\,\frac{1}{1+\xi^2} - \frac{4P}{\pi}\,\frac{1-\xi^2}{(1+\xi^2)^2}$

$\omega''(\xi) = -\frac{8P}{\pi}\,\frac{1}{(1+\xi^2)^2}$

Also from eqns. (6) $\qquad A_1 = -\bar{A}_1 - \frac{4P}{\pi}$

Real part of $A_1 = \frac{A_1+\bar{A}_1}{2} = -\frac{2P}{\pi}$

but as A_1 is real $\qquad A_1 = -\frac{2P}{\pi}$

so from eqn. (5)

$\Omega'(\xi) = -\bar{A}_1 - \frac{4P}{\pi}\left(\frac{1}{1+\xi^2}\right)$

$\Omega'(\xi) = -\frac{2P}{\pi}\left(\frac{1-\xi^2}{1+\xi^2}\right)$

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e.g. $\quad z=0 \qquad \widehat{xx}+\widehat{yy} = -\frac{2P}{\pi}$
$(\xi=0)$

$\widehat{xx}-\widehat{yy}-2i\widehat{xy} = -\frac{1}{2}(\bar{z}\,\Omega''+\omega'')$

$= -\frac{1}{2}\omega'' = \frac{4P}{\pi}$

so $\widehat{xy}=0$

$\widehat{xx}=\frac{P}{\pi}$

$\widehat{yy}=-\frac{3P}{\pi}$

$$z = iy \qquad \widehat{xx}+\widehat{yy} = -\frac{2P}{\pi}\,\frac{1+y^2}{1-y^2}$$

$$(\xi = iy)$$

$$\widehat{xx}-\widehat{yy}-2i\widehat{xy} = -\frac{1}{2}\left(\frac{8P}{\pi}\,\frac{\xi\bar{\xi}}{(1+\xi^2)^2}-\frac{8P}{\pi}\,\frac{1}{(1+\xi^2)^2}\right)$$

$$= -\frac{1}{2}\left(\frac{8P}{\pi}\,\frac{y^2}{(1-y^2)^2}-\frac{8P}{\pi}\,\frac{1}{(1-y^2)^2}\right)$$

$$= -\frac{1}{2}\,\frac{8P}{\pi}\,\frac{1}{y^2-1}$$

$$= \frac{4P}{\pi}\,\frac{1}{1-y^2}$$

$$\widehat{xy} = 0$$

$$\widehat{xx} = \frac{P}{\pi}$$

$$\widehat{yy} = -\frac{P}{\pi}\,\frac{y^2+3}{1-y^2}$$

$$z = x \longrightarrow \qquad \widehat{xx} = \frac{P}{\pi}\,\frac{(1-x^2)^2}{(1+x^2)^2}$$

$$\widehat{yy} = \frac{P}{\pi}\,\frac{(x^2+3)(x^2-1)}{(1+x^2)^2}$$

$$\widehat{xy} = 0$$

The previous problem was artifically simple, however, because of the simplicity of the shape chosen i.e. the unit circle. To investigate the stresses in more complex shapes it is necessary to transform them, by a conformed transformation, onto the unit circle.

SECTION 3

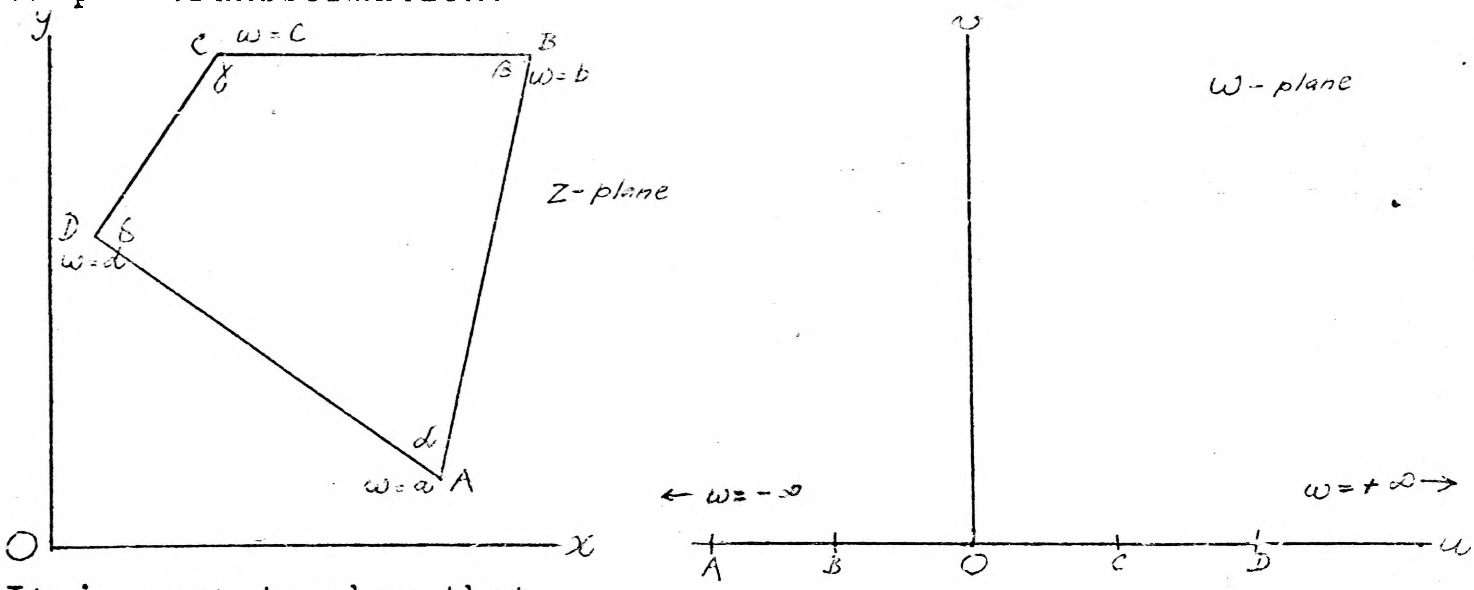
DERIVATION OF THE INFINITE SET OF SIMULTANEOUS EQUATIONS

SECTION 3.1

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

It is not always possible to find a mapping function which transforms the required shape in the z-plane to the unit circle in the ξ-plane, but in the case of a closed polygon this can be done and the transformation is due to Schwarz and Christoffel.

The Schwarz-Christoffel transformation maps the interior of any closed polygon onto the upper half of the ω-plane. The ω-plane can then be mapped onto the unit circle by a simple transformation.



It is easy to show that

(ref. 6 p. 193 seq.)

$$\frac{dz}{d\omega} = A(\omega-a)^{\alpha/\pi-1}\,(\omega-b)^{\beta/\pi-1}\,(\omega-c)^{\gamma/\pi-1}\ldots$$

where A is a constant, $\alpha, \beta\ \gamma, \ldots$ are the interior angles of the polygon and a,b,c,... are the values the corresponding points on the polygon have when mapped onto the real axis of the ω-plane.

SECTION 3.2

APPLICATION OF SCHWARZ-CHRISTOFFEL TRANSFORMATION TO THEORY OF ELASTICITY

Section 3.2.1 Derivation of left hand sides of simultaneous equations

Reverting to our boundary equation

$$\Omega(\sigma) + f(\sigma)\overline{\Omega}_z'(\sigma) + \overline{\omega}_\sigma'(\overline{\sigma}) = F_1 + iF_2 \qquad (3.1),$$

we wish to find Ω_z' and ω_z'

where $\Omega_z' = \frac{d\Omega}{dz}$

and $z = f(\xi)$ transforms z to circle $|\xi| \leq 1$

Let $\omega' = a_0 + a_1\xi + a_2\xi^2 + \ldots \qquad |\xi| < 1$

Then using Cauchy's integral, we obtain

$$\frac{1}{2\pi i}\int_\gamma \overline{\omega}'(\sigma)\frac{d\sigma}{\sigma-\xi} = \overline{a}_0 \qquad \text{where } \gamma \text{ is a contour of integration}$$

and equation (3.1) above becomes

$$\Omega(\xi) = \frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi} - \frac{1}{2\pi i}\int_\gamma z\,\overline{\Omega}'\frac{d\sigma}{\sigma-\xi} - \overline{a}_0$$

Differentiating with respect to ξ gives

$$\Omega'(\xi) = \left[\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi}\right]' - \frac{d}{d\xi}\left(\frac{1}{2\pi i}\int_\gamma z\,\overline{\Omega}_z'\frac{d\sigma}{\sigma-\xi}\right)$$

Assume $\quad \frac{d}{d\xi}\left(\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi}\right) = \sum_{r=0}^{\infty} P_r\xi^r$

and the equation becomes

$$\Omega'(z)\frac{dz}{d\xi} = \sum_{r=0}^{\infty} P_r\xi^r - \frac{d}{d\xi}\left(\frac{1}{2\pi i}\int_\gamma z\,\overline{\Omega}_z'\frac{d\sigma}{\sigma-\xi}\right) \quad \ldots \quad (3.2)$$

Let $\Omega'(z) = b_0 + b_1\xi + b_2\xi^2 + \ldots$

and $z = f(\xi) = u_0 + u_1\xi + u_2\xi^2 + \ldots$

then $\frac{dz}{d\xi} = u_1 + 2u_2\xi + 3u_3\xi^2 + \dots$

and $\frac{1}{2\pi i}\int_\gamma Z\overline{\Omega}_z' \frac{d\sigma}{\sigma-\xi} = \frac{1}{2\pi i}\int_\gamma (u_0 + u_1\sigma + u_2\sigma^2 + \dots)\left(\overline{b}_0 + \frac{\overline{b}_1}{\sigma} + \frac{\overline{b}_2}{\sigma^2} + \dots\right)\frac{d\sigma}{\sigma-\xi}$

$= (u_0\overline{b}_0 + u_1\overline{b}_1 + u_2\overline{b}_2 + \dots) + \xi(u_1\overline{b}_0 + u_2\overline{b}_1 + u_3\overline{b}_2 + \dots)$

$+ \xi^2(u_2\overline{b}_0 + u_3\overline{b}_1 + u_4\overline{b}_2 + \dots) + \dots$ since only positive powers of ξ remain and

$\frac{d}{d\xi}\left[\frac{1}{2\pi i}\int_\gamma Z\overline{\Omega}_z' \frac{d\sigma}{\sigma-\xi}\right]$

$= (u_1\overline{b}_0 + u_2\overline{b}_1 + u_3\overline{b}_2 + \dots)$

$+2(u_2\overline{b}_0 + u_3\overline{b}_1 + u_4\overline{b}_2 + \dots)\xi$

$3(u_3\overline{b}_0 + u_4\overline{b}_1 + u_5\overline{b}_2 + \dots)\xi^2 + \dots$

Equation (3.2) becomes

$(b_0 + b_1\xi + b_2\xi^2 + \dots)(u_1 + 2u_2\xi + 3u_3\xi^2 + \dots)$

$= \sum_{r=0}^{\infty} P_r\xi^r - (u_1\overline{b}_0 + u_2\overline{b}_1 + u_3\overline{b}_2 + \dots)$

$-2(u_2\overline{b}_0 + u_3\overline{b}_1 + u_4\overline{b}_2 + \dots)\xi - 3(u_3\overline{b}_0 + u_4\overline{b}_1 + \quad)\xi^2$

$- \dots$

coeff. of ξ^0 gives

$$u_1 b_0 + (u_1\overline{b}_0 + u_2\overline{b}_1 + u_3\overline{b}_2 + \dots) = P_0 \qquad (3.3)$$

$$\xi^1 \dots u_1 b_1 + 2u_2 b_0 + 2(u_2\overline{b}_0 + u_3\overline{b}_1 + u_4\overline{b}_2 + \dots) = P_1 \qquad (3.4)$$

$$\xi^2 \dots u_1 b_2 + 2u_2 b_1 + 3u_3 b_0 + 3(u_3\overline{b}_0 + u_4\overline{b}_1 + \dots) = P_2 \qquad (3.5)$$

$$\xi^3 \dots u_1 b_3 + 2u_2 b_2 + 3u_3 b_1 + 4u_4 b_0 + 4(u_4\overline{b}_0 + u_5\overline{b}_1 + u_6\overline{b}_2 + \dots) = P_3 \qquad (3.6)$$

and so on

To find ω', take the conjugate of the boundary equation

i.e.

Again using Cauchy's integral gives

$$\omega'(\xi) = \frac{1}{2\pi i}\int_\gamma (F_1 - iF_2)\frac{d\sigma}{\sigma-\xi} - \frac{1}{2\pi i}\int_\gamma \bar{f}(\bar\sigma)\Omega'(\sigma)\frac{d\sigma}{\sigma-\xi}$$
$$- \frac{1}{2\pi i}\int_\gamma \bar\Omega(\sigma)\frac{d\sigma}{\sigma-\xi}$$

Now $\Omega'_z = b_0 + b_1\xi + b_2\xi^2 + \dots$

so $\Omega_z = \int \Omega'_z dz$

$$= \int (b_0 + b_1\xi + b_2\xi^2 + \dots)(u_1 + 2u_2\xi + 3u_3\xi^2 + \dots)d\xi$$

$$= \int (b_0u_1 + \xi(b_1u_1 + 2b_2u_2) + \xi^2(b_2u_1 + 2b_1u_2 + 3b_0u_3) + \dots)d\xi$$

$$= C + b_0u_1\xi + \tfrac{1}{2}(b_1u_1 + 2b_0u_2)\xi^2 + \dots$$

So $\frac{1}{2\pi i}\int \Omega \frac{d\sigma}{\sigma-\xi} = \bar{C}$

Then $$\frac{d}{d\xi}(\omega'(\xi)) = \sum_{r=0}^{\infty} Q_r\xi^r - \frac{d}{d\xi}\left(\frac{1}{2\pi i}\int \bar{Z}\Omega'_z\frac{d\sigma}{\sigma-\xi}\right) \quad \text{----} (3.7)$$

using $\frac{1}{2\pi i}\int_\gamma (F_1 - iF_2)\frac{d\sigma}{\sigma-\xi} = \sum_{r=0}^{\infty} Q_r\xi^r$

Now $\frac{1}{2\pi i}\int_\gamma (F_1-iF_2)\frac{d\sigma}{\sigma-\xi}$ = positive power terms of

$$\left[\left(\bar{u}_0 + \frac{\bar{u}_1}{\xi} + \frac{\bar{u}_2}{\xi^2} + \dots\right)(b_0 + b_1\xi + b_2\xi^2 + \dots)\right]$$

$$= (\bar{u}_0 b_0 + \bar{u}_1 b_1 + \bar{u}_2 b_2)$$

$$+ \xi(\bar{u}_0 b_1 + \bar{u}_1 b_2 + \dots)$$

$$+ \xi^2(\bar{u}_0 b_2 + \bar{u}_1 b_3 + \dots)$$

and $\frac{d}{d\xi}\left[\frac{1}{2\pi i}\int \bar{Z}\Omega'_z \frac{d\sigma}{\sigma-Z}\right]$

$$= (\bar{u}_0 b_1 + \bar{u}_1 b_2 + \bar{u}_2 b_3 + \dots)$$

$$+ 2\xi(\bar{u}_0 b_2 + \bar{u}_1 b_3 + \dots)$$

$$+ 3\xi^2(\bar{u}_0 b_3 + \bar{u}_1 b_4 + \dots)$$

$$+ \dots$$

Equation (3.7) becomes

$$a_1 + 2a_2\xi + 3a_3\xi^2 + \ldots = \sum_{r=0}^{\infty} Q_r \xi^r - (\bar{u}_0 b_1 + \bar{u}_1 b_2 + \ldots)$$
$$-2(\bar{u}_0 b_2 + \bar{u}_1 b_3 + \ldots)\xi - \ldots$$

Coeff of ξ^0 gives $\quad a_1 + \bar{u}_0 b_1 + \bar{u}_1 b_2 + \ldots = Q_0 \quad \ldots \quad (3.8)$

$\xi^1 \quad 2a_2 + 2(\bar{u}_0 b_2 + \bar{u}_1 b_3 + \ldots) = Q_1 \quad \ldots \quad (3.9)$

$\xi^2 \quad 3a_3 + 3(\bar{u}_0 b_3 + \bar{u}_1 b_4 + \ldots) = Q_2 \quad \ldots \quad (3.10)$

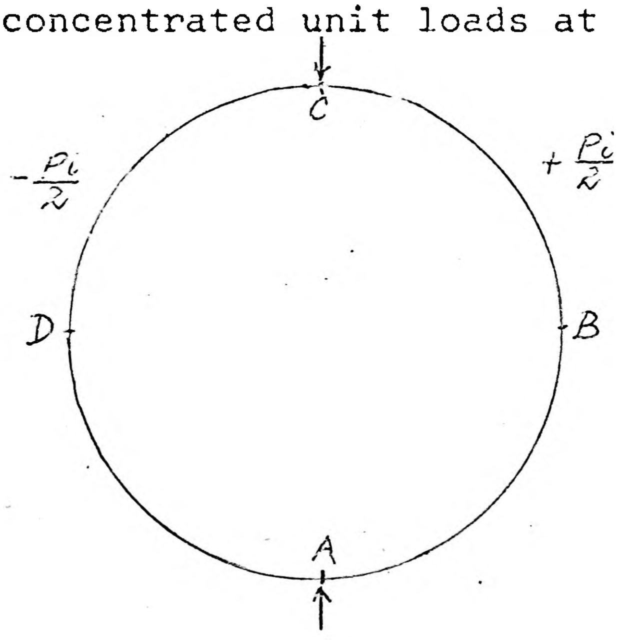
etc.

Once the b's have been found by solving the infinite set of simultaneous equations with infinite unknowns (eqns. 3.3 to 3.6) the a's can be found by direct substitution in these eqns., after calculating the Q's.

Section 3.2 2 calculation of right hand sides in a particular case.

To find P_0, P_1, P_2, ...

take the typical case where the unit circle has concentrated unit loads at +i and -i



The value of $F_1 + iF_2$ can be taken as $\frac{-Pi}{2}$ .4i

on CDA and $+\frac{Pi}{2}$ .4i on ABC.

Then $\frac{1}{2\pi i}\int (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi}$

$$= \frac{1}{2\pi i}\left[\int_{\sigma_a}^{\sigma_c} -\frac{2Pd\sigma}{\sigma-\xi} + \int_{\sigma_c}^{\sigma_a}\frac{2Pd\sigma}{\sigma-\xi}\right]$$

then $\frac{d}{d\xi}\left[\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi}\right]$

$$= -\frac{2iP}{\pi}\left[\frac{1}{i+\xi} + \frac{1}{i-\xi}\right]$$ since $\sigma_c = +i$ and $\sigma_a = -i$

$$= -\frac{4P}{\pi}\,\frac{1}{1+\xi^2}$$

$$= -\frac{4P}{\pi}\left(1-\xi^2 + \xi^4 - \xi^6 + \xi^8 - \dots\right)$$

But $\frac{d}{d\xi}\left[\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma-\xi}\right] = \sum_{r=0}^{\infty} P_r\xi^r$ by convention,

so $P_0 = -\frac{4P}{\pi}$, $P_2 = \frac{4P}{\pi}$, $P_4 = -\frac{4P}{\pi}, \dots$

$P_1 = P_3 = P_5 = P_7 = \dots = 0$

Note that since the P's are real $Q_n = P_n$.

SECTION 4

SOLUTIONS OF EQUATIONS OF SIMILAR TYPE TO THOSE ENCOUNTERED IN THE ELASTICITY THEORY.

SECTION 4.1

TYPE OF EQUATIONS INVESTIGATED

An example of the type of equations that equations 3.3 to 3.6 become is;

$$x_1 + x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \ldots = 1 \tag{4.1}$$

$$x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \ldots = -1 \tag{4.2}$$

$$x_3 + \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \ldots = 1 \tag{4.3}$$

$$x_4 + \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 + \ldots = -1 \tag{4.4}$$

etc.

and this set of equations shall be used to give an example of two methods used in the sequel to solve the equations obtained.

No success has been obtained in trying to extend the theory of convergence by a successive approximation method in the finite case to the infinite case.

All that has been attempted, when deriving equations, is to make the coefficients of the diagonal elements predominate in their respective rows and to make the right hand sides as small as possible, especially as the equation number approaches infinity.

SECTION 4.2

METHOD 1 ("PURE ITERATION")

Section 4.2.1 Theoretical method.

Find x_1 from equation 4.1, letting $x_2, x_3, \ldots = 0$; then find x_2 from equation 4.2, using the value found for x_1 but letting $x_3, x_4, \ldots$ remain equal to 0. Then go back to equation 4.1 and find x_1, letting the other x's remain constant, find x_2 from equation 4.2 and then x_3 from equation 4.3. Then find x_1 from equation 4.1, x_2 from equation 4.2, x_3 from equation 4.3, x_4 from equation 4.4, then loop back to equation 1.

Section 4.2.2 Discussion of computer programme used and results.

A computer programme written in FORTRAN (PROG1) is shown. The values of each x_r oscillated between slowly 'converging' limits as each new loop was performed.

e.g. on the loop which calculated x_{48}, the programme calculated $x_1=.6764313$, $x_2=-1.1088249$, $x_3=.943647$, $x_4=-1.0354112$, $x_5=.9751291, \ldots$ $x_{45}=.9983780$, $x_{46}=-1.0015819$, $x_{47}=.9983453$, $x_{48}=-.9912218$.

and on the loop which calculated x_{49},

$x_1=.7093965$, $x_2=-1.0846967$, $x_3=.9636028$, $x_4=-1.0179970$, $x_5=.9907800, \ldots$ $x_{45}=1.0023050$, $x_{46}=-.9977354$, $x_{47}=1.0022255$, $x_{48}=-.9977052$, $x_{49}=-.9920676$

and on the loop which calculated x_{50},
$x_1=.6768491$, $x_2=-1.1085255$, $x_3=.9438896$, $x_4=-1.0352043$, $x_5=.9753102$,... $x_{45}=.9983801$, $x_{46}=-1.0015817$, $x_{47}=.9984566$, $x_{48}=-1.0015065$, $x_{49}=.9984259$, $x_{50}=-.9915599$.

Obviously $x_n \to \pm 1$, as $n \to \infty$,
but the 'convergence' of the oscillation of the x's is somewhat slow. See APPENDIX A Sect.1 and the computer listing of PROG1 for further details.

The second method is designed to give an accurate result more quickly.

SECTION 4.3

METHOD 2 ("SUCCESSIVE APPROXIMATION")

Section 4.3.1 Theoretical method.

In this case let $x_1=1$, $x_2=-1$, $x_3=1$, $x_4=-1$,... and evaluate the left hand sides of the equations. Then adjust x_1 to make equation 4.1 hold, use the new value of x_1 in equation 4.2 and calculate the change in x_2 to make equation 4.2 hold. Using this change in x_2, calculate the new change in x_1 required to make equation 4.1 hold and use this new value to find the new change in x_2 required to make equation 4.2 hold. Then calculate the change in x_3 required to make equation 4.3 hold and loop as before.

The details are as follows:

Call ΔR_1, ΔR_2, ΔR_3, etc. the amounts that must be added to the right hand sides R_1, R_2, R_3, etc. to give the values 1, -1, 1, -1,... Then adjust x_1 to add ΔR_1 to R_1. Adjust x_2 in equation 4.2 to add ΔR_2 to R_2, noting that the change in x_1 affects R_2. Then go back and adjust x_1 taking into account the new value of x_2, then adjust x_2 considering the new value of x_1.

Then adjust x_3 in equation 4.3 to add the appropriate amount to R_3 to make equation 4.3 hold. Then readjust x_1, x_2, x_3 and go on and calculate the adjustment required of x_4, and so on.

The evaluation of the l.h.s's of the equations was simplein this case but can be extremely difficult as will be seen later.

l.h.s. of equation 4.1

$$=1 + 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \dots$$

$$=1 + \log_e 2 \text{ using } \log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots$$

So $-\log_e 2$ has to be added to it to make it equal the r.h.s.

Thus $\underline{\Delta R_1 = -\log_e 2}$

l.h.s. of equation 4.2

$$=-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \dots$$

$$= -\log_e 2$$

so $\underline{\underline{\Delta R_2 = -1 + \log_e 2}}$

l.h.s. of equation 4.3

$$=1 + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots$$

$$= \tfrac{1}{2} + \log_e 2$$

$$\underline{\underline{\Delta R_3 = \tfrac{1}{2} - \log_e 2}}$$

and so on.

Section 4.3.2 Discussion of computer programme used and results.

A FORTRAN programme PRO2 solves the equations by this method.

The convergence was very good, Δx_1 (the amount needed to be added to the initial value of x_1) being -.30515 at 15 terms, a maximum of -.30508 around 23 terms, and -.30515 at 40 terms. Other x's converged similarly. Note that the value each x converged to is the mid-point of the oscillation in the 'pure iteration' method.

For further details see the computer listing PRO2 and APPENDIX A Sect.2.

SECTION 4.4

TWO OTHER SETS OF EQUATIONS SOLVED BY SUCCESSIVE APPROXIMATION.

$$x_1 + x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_2 + \ldots = 1$$

$$x_2 + \frac{1}{2^2} x_1 + \frac{1}{3^2} x_2 + \frac{1}{4^2} x_3 + \ldots = 1$$

$$x_3 + \frac{1}{3^2} x_1 + \frac{1}{4^2} x_2 + \frac{1}{5^2} x_3 + \ldots = 1$$

and

$$x_1 + x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \ldots = 1$$

$$x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \ldots = 0$$

$$x_3 + \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \ldots = -1$$

$$x_4 + \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 + \ldots = 0$$

$$x_5 + \tfrac{1}{5}x_1 + \tfrac{1}{6}x_2 + \tfrac{1}{7}x_3 + \ldots = 1$$

$$x_6 + \tfrac{1}{6}x_1 + \tfrac{1}{7}x_2 + \tfrac{1}{8}x_3 + \ldots = 0$$

were also solved using method 2.

For further details see APPENDIX A Sects.3 & 4.

SECTION 5

SOLUTION OF Ω_z' FOR BEAM UNDER PARTICULAR LOADING USING SUCCESSIVE APPROXIMATION

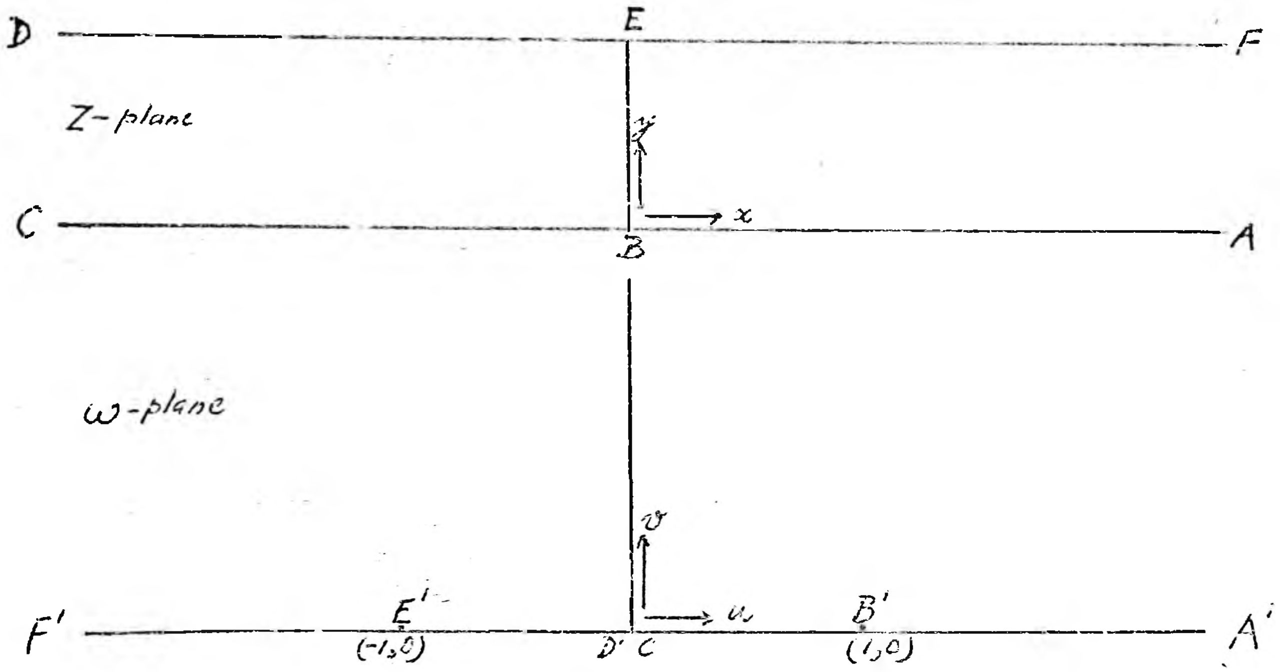
SECTION 5.1

DERIVATION OF EQUATIONS

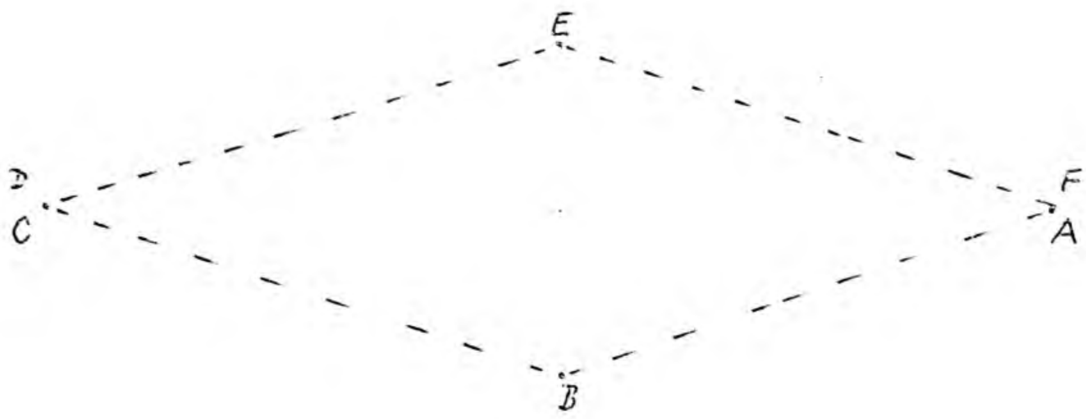
Section 5.1.1 Derivation of Schwarz-Christoffel Transformation.

For a first example of this method applied to the solution of an actual problem, consider an infinitely long beam of width π with point compressive loads applied normally to opposing points on the sides.

If the width is any other value, say a, the values of the stresses will be $\frac{\pi}{a}$ times those calculated here.



Although the beam is infinitely long we can consider it as a closed polygon which comes together at ∞.



Then it can be opened out at C.

$-\infty$ on the z-plane coincides with 0 on the ω-plane.

$+\infty$ on the z-plane coincides with ∞ on the ω-plane.

$\infty + i\pi$ on the z-plane coincides with $-\infty$ on the ω-plane.

Points E and B on the z-plane can be arbitrarily given values of -1 and +1 respectively on the ω-plane.

Then $\frac{dz}{d\omega} = A(\omega - 0)^{\frac{0}{\pi}-1}$ since the interior angle at C and D = 0 and the corresponding point on the ωplane is 0.

so $\frac{dz}{d\omega} = \frac{A}{\omega}$

$Z = A\log\omega$

To find the constant A :

for $Z = i\pi$ $\omega = -1$

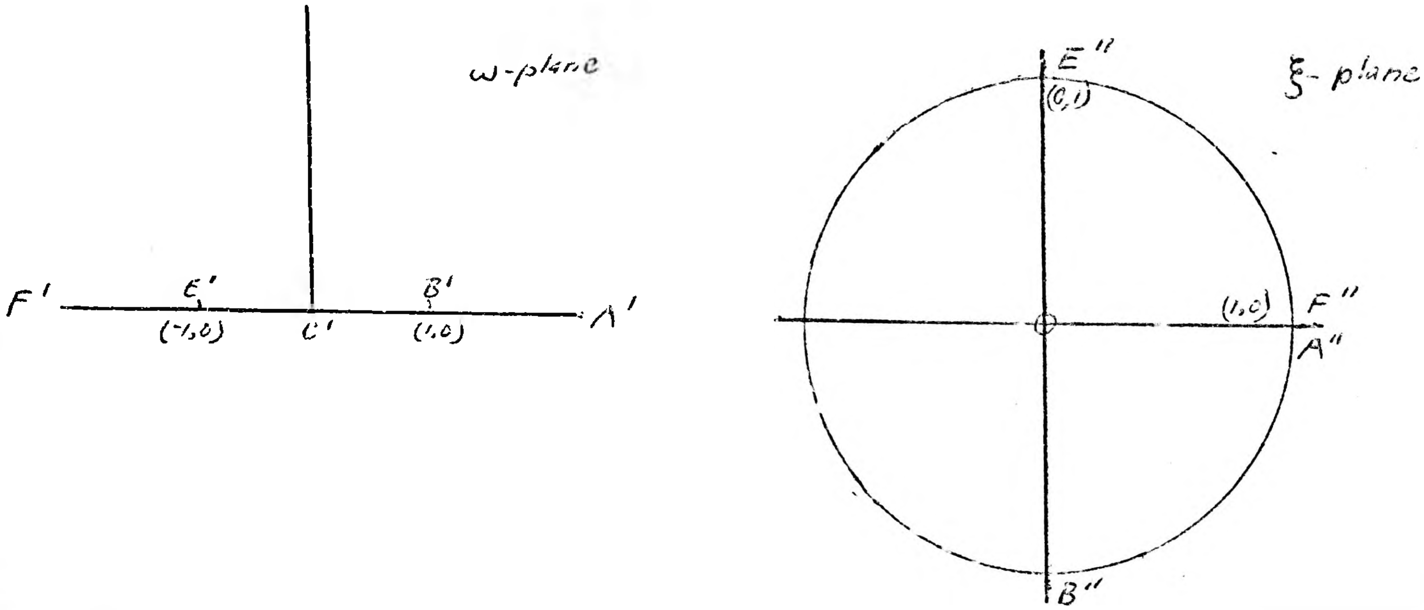
then $\log\omega = \log(-1) = i\pi$

so $A = 1$

[If width is a , $A = \frac{a}{\pi}$

and $Z = \frac{a}{\pi}\log\omega$]

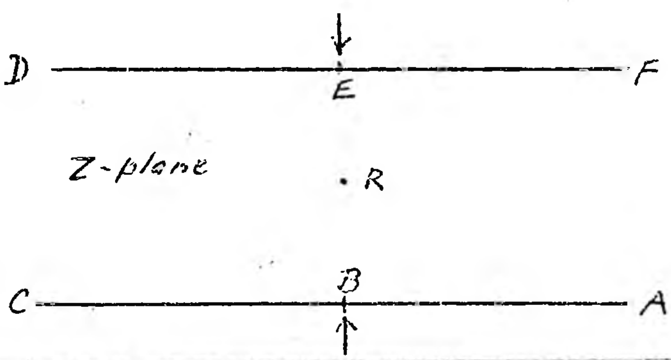
It is now required to make a further transformation of ω onto the unit circle.

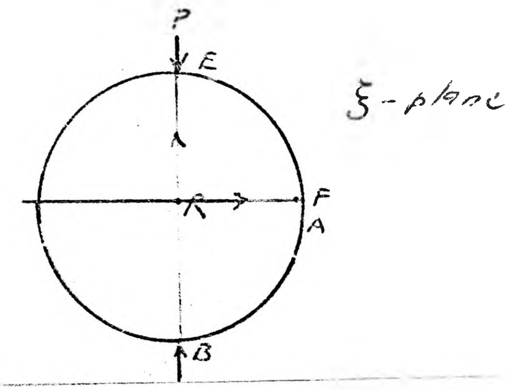


The transformation is

$$\omega = i\,\frac{1+\xi}{1-\xi}$$

so $z = \log i\,\frac{1+\xi}{1-\xi}$ maps the beam onto the unit circle as shown





<u>Section 5.1.2</u> <u>Derivation of infinite set of simultaneous equations.</u>

$$z = \log i\,\frac{1+\xi}{1-\xi}$$

$$= \frac{i\pi}{2} + \left(\xi - \frac{\xi^2}{2} + \frac{\xi^3}{3} - \dots\right) - \left(-\xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} - \dots\right)$$

$$z = i\frac{\pi}{2} + 2\left(\xi + \frac{\xi^3}{3} + \frac{\xi^5}{5} + \dots\right)$$

so, if $z = u_0 + u_1\xi + u_2\xi^2 + \dots$

$$u_0 = \frac{i\pi}{2}, \quad u_1 = 2, \quad u_3 = \frac{2}{3}, \quad u_5 = \frac{2}{5}, \quad \ldots$$

$$u_2 = u_4 = u_6 = \ldots = 0$$

If we take the beam loaded as shown, this corresponds to the loading shown on the unit circle, for which we have previously found

$$P_0 = -\frac{4P}{\pi}, \quad P_2 = \frac{4P}{\pi}, \quad P_4 = -\frac{4P}{\pi}, \quad \ldots$$

$$P_1 = P_3 = P_5 = \ldots = 0$$

So eqns. 3.3 to 3.6 on p 14 become

$$2\left(b_0 + (\bar{b}_0 + \tfrac{1}{3}\bar{b}_2 + \tfrac{1}{5}\bar{b}_4 + \ldots\quad)\right) = -\frac{4P}{\pi}$$

$$2\left(b_1 + 2(\tfrac{1}{3}\bar{b}_1 + \tfrac{1}{5}\bar{b}_3 + \tfrac{1}{7}\bar{b}_5 + \ldots\quad)\right) = 0$$

$$2\left(b_2 + 3.\tfrac{1}{3}b_0 + 3(\tfrac{1}{3}\bar{b}_0 + \tfrac{1}{5}\bar{b}_2 + \tfrac{1}{7}\bar{b}_4 + \ldots\quad)\right) = \frac{4P}{\pi}$$

$$2\left(b_3 + 3.\tfrac{1}{3}b_1 + 4(\tfrac{1}{5}\bar{b}_1 + \tfrac{1}{7}\bar{b}_3 + \ldots\quad)\right) = 0$$

$$2\left(b_4 + 3.\tfrac{1}{3}b_2 + 5.\tfrac{1}{5}b_0 + 5(\tfrac{1}{5}\bar{b}_0 + \tfrac{1}{7}\bar{b}_2 + \ldots\quad)\right) = -\frac{4P}{\pi}$$

and so on.

Let $b_n = \alpha_n + i\beta_n$, then the equations of the imaginary coefficients are all satisfied by $\beta_n = 0$ for all n, and the equations of the real coeffs. are :-

$$\alpha_0 + \qquad (\alpha_0 + \tfrac{1}{3}\alpha_2 + \tfrac{1}{5}\alpha_4 + \ldots\quad) = -\frac{2P}{\pi}$$

$$\alpha_1 + \qquad 2(\tfrac{1}{3}\alpha_1 + \tfrac{1}{5}\alpha_3 + \tfrac{1}{7}\alpha_5 + \ldots\quad) = 0$$

$$\alpha_2 + \alpha_0 + \qquad 3(\tfrac{1}{3}\alpha_0 + \tfrac{1}{5}\alpha_2 + \tfrac{1}{7}\alpha_4 + \ldots\quad) = \frac{2P}{\pi}$$

$$\alpha_3 + \alpha_1 + \qquad 4(\tfrac{1}{5}\alpha_1 + \tfrac{1}{7}\alpha_3 + \tfrac{1}{9}\alpha_5 + \ldots\quad) = 0$$

$$\alpha_4 + \alpha_2 + \alpha_0 + \qquad 5(\tfrac{1}{5}\alpha_0 + \tfrac{1}{7}\alpha_2 + \tfrac{1}{9}\alpha_4 + \ldots\quad) = -\frac{2P}{\pi}$$

$$\alpha_5 + \alpha_3 + \alpha_1 + \qquad 6(\tfrac{1}{7}\alpha_1 + \tfrac{1}{9}\alpha_3 + \tfrac{1}{11}\alpha_5 + \ldots\quad) = 0$$

and so on.

The equation obtained by subtracting eqn. (5.I) from eqn. (5.3) is given the name "(EQN Δ3)", the equation obtained by subtracting eqn. (5.3) from eqn. (5.5) is given the name "(EQN Δ5)", and so on.

$\alpha_1 = \alpha_3 = \alpha_5 = \ldots = 0$ satisfies the eqns..

Letting $\alpha_0 = \frac{2P}{\pi}x_1$, $\alpha_1 = \frac{2P}{\pi}x_2$, $\alpha_2 = \frac{2P}{\pi}x_3$, ...

gives

$$(5.1) \qquad x_1 + (x_1 + \tfrac{1}{3}x_3 + \tfrac{1}{5}x_5 + \ldots) = -1$$

$$(5.3) \qquad x_3 + x_1 + 3(\tfrac{1}{3}x_1 + \tfrac{1}{5}x_3 + \tfrac{1}{7}x_5 + \ldots) = 1$$

$$(5.5) \qquad x_5 + x_3 + x_1 + 5(\tfrac{1}{5}x_1 + \tfrac{1}{7}x_3 + \tfrac{1}{9}x_5 + \ldots) = -1$$

$$(5.7) \qquad x_7 + x_5 + x_3 + x_1 + 7(\tfrac{1}{7}x_1 + \tfrac{1}{9}x_3 + \ldots) = 1$$

and so on.

Then, taking eqn.(5.3) - eqn.(5.1), eqn.(5.5) - eqn.(5.3), and so on gives

$$(5.3)-(5.1) \quad (\text{EQN } \Delta 3) \qquad x_3 + x_3(\tfrac{3}{5} - \tfrac{1}{3}) + x_5(\tfrac{3}{7} - \tfrac{1}{5}) + \ldots = 2$$

$$(5.5)-(5.3) \quad (\text{EQN } \Delta 5) \qquad x_5 + x_3(\tfrac{5}{7} - \tfrac{3}{5}) + x_5(\tfrac{5}{9} - \tfrac{3}{7}) + \ldots = -2$$

$$(5.7)-(5.5) \quad (\text{eqn } \Delta 7) \qquad x_7 + x_3(\tfrac{7}{9} - \tfrac{5}{7}) + x_5(\tfrac{7}{11} - \tfrac{5}{9}) + \ldots = 2$$

$$(5.9)-(5.7) \quad (\text{EQN } \Delta 9) \qquad x_9 + x_3(\tfrac{9}{11} - \tfrac{7}{9}) + x_5(\tfrac{9}{13} - \tfrac{7}{11}) + \ldots = -2$$

and so on.

SECTION 5.2

CALCULATION OF Ω'_z

Section 5.2.1 Calculation of coefficients of Ω'_z.

To solve these equations using method 2, let

$$x_3 = 2, \quad x_5 = -2, \quad x_7 = 2, \quad x_9 = -2, \quad \ldots$$

then l.h.s's of eqns. $(\Delta 3)$, $(\Delta 5)$, $(\Delta 7)$, ... are

$$(\Delta 3) \qquad 2\left(1 + (\tfrac{3}{5} - \tfrac{1}{3}) - (\tfrac{3}{7} - \tfrac{1}{5}) - (\tfrac{3}{9} - \tfrac{1}{7}) + \ldots\right) = C_3$$

$$(\Delta 5) \qquad 2\left(-1 + (\tfrac{5}{7} - \tfrac{3}{5}) - (\tfrac{5}{9} - \tfrac{3}{7}) - (\tfrac{5}{11} - \tfrac{3}{9}) + \ldots\right) = C_5$$

$$(\Delta 7) \qquad 2\left(1 + (\tfrac{7}{9} - \tfrac{5}{7}) - (\tfrac{7}{11} - \tfrac{5}{9}) + (\tfrac{7}{13} - \tfrac{5}{11}) + \ldots\right) = C_7 \text{ etc.}$$

$$C_3 = 2 + 2\left(3\left(\tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} \ldots\right) - 1\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{9} \ldots\right)\right)$$

$$= 2 + 2\left(3\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3}}\right) - \left(1 - \tfrac{\pi}{4}\right)\right)$$

$$C_5 = -2 + 2\left(5\left(\tfrac{1}{7} - \tfrac{1}{9} + \tfrac{1}{11} \ldots\right) - 3\left(\tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \quad \ldots\right)\right)$$

$$= -2 + 2\left(5\left(\overline{1 - \tfrac{1}{3} + \tfrac{1}{5}} - \tfrac{\pi}{4}\right) - 3\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3}}\right)\right)$$

$$C_7 = 2 + 2\left(7\left(\tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} \ldots\right) - 5\left(\tfrac{1}{7} - \tfrac{1}{9} + \tfrac{1}{11} - \quad \ldots\right)\right)$$

$$= 2 + 2\left(7\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}}\right) - 5\left(\overline{1 - \tfrac{1}{3} + \tfrac{1}{5}} - \tfrac{\pi}{4}\right)\right)$$

and so on.

So, if ΔC_r is the amount that needs to be added to C_r to make it equal to $+2$ or -2,

$$\Delta C_3 = -2\left[3\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3}}\right) - \left(1 - \tfrac{\pi}{4}\right)\right] \qquad \simeq -2 \times .1418$$

$$\Delta C_5 = -2\left[5\left(\overline{1 - \tfrac{1}{3} + \tfrac{1}{5}} - \tfrac{\pi}{4}\right) - 3\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3}}\right)\right] \qquad \simeq -2 \times .0506$$

$$\Delta C_7 = -2\left[7\left(\tfrac{\pi}{4} - \overline{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}}\right) - 5\left(\overline{1 - \tfrac{1}{3} + \tfrac{1}{5}} - \tfrac{\pi}{4}\right)\right]$$

$$\simeq -2 \times .0252$$

The amount by which the x_r are required to change (DX(R)) was calculated in the same manner as before by the FORTRAN programme BEAM2. i.e. x_3, x_5, x_7, were calculated.Then equation 5·1 was used to find x_1.

The following results were obtained:

DX (3) = -.20328

DX (5) = -.06194

DX(7) = -.02653

DX (9) = -.01352

DX (11)= -.00786

DX (13)= -.00474

DX (15)= -.00325 etc.

Again, the convergence was very fast.

This gave $x_1 = -.67105$ calculated from eqn. 5.1.

The values obtained are thus

$x_1 = -.67105$ $\qquad$ $x_3 = 1.7969$

$x_5 = -2.0620$ $\qquad$ $x_7 = 1.9735$

$x_9 = -2.0135$ $\qquad$ $x_{11} = 1.9921$

$x_{13} = -2.0047$ $\qquad$ etc.

then
$$\alpha_0 = -.67105 \times \frac{2P}{\pi} = b_0$$
$$\alpha_2 = 1.7967 \times \frac{2P}{\pi} = b_2$$
$$\alpha_4 = -2.0619 \times \frac{2P}{\pi} = b_4$$
$$\alpha_6 = 1.9735 \times \frac{2P}{\pi} = b_6 \qquad \text{etc.}$$

and $b_1 = b_3 = b_5 = \ldots = 0$

This gives Ω'_z where

$$\Omega'_z = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \ldots$$

See APPENDIX B for extra details on programme BEAM2.

Section 5.2.2 Splitting of infinite series into two more-manageable series.

$$\Omega'_z = b_0 + b_1\xi + b_2\xi^2 + \ldots$$
$$= \frac{2P}{\pi}\Big(-.67105 + (2 - .20328)\xi^2 + (-2 - .06194)\xi^4 + (+2 - .02653)\xi^6 + \ldots\Big)$$
$$= \frac{2P}{\pi}\Big(-2 + 2\xi^2 - 2\xi^4 + 2\xi^6 - \ldots\Big) + \frac{2P}{\pi}\Big(1.32895 - (.20328\xi^2 + .06194\xi^4 + .02653\xi^6 + .01352\xi^8 + .00786\xi^{10} + .00474\xi^{12} + \ldots)\Big)$$
$$\Omega'_z = -\frac{4P}{\pi}\cdot\frac{1}{1+\xi^2} + \frac{2P}{\pi}\Big(1.32895 - (.20328\xi^2 + .06194\xi^4 + .02653\xi^6 + .01352\xi^8 + .00786\xi^{10} + \ldots)\Big)$$

Section 5.2.3 Stress calculation at critical points.

When $\xi = \pm 1$ i.e. $z = \pm\infty$,

the above formula gives Ω'_z very nearly equal to 0 as expected.

When $\xi = 0$ i.e. $z = i\frac{\pi}{2}$

$$\Omega'_z = -\frac{4P}{\pi}\left(1 - \tfrac{1}{2}(1.32895)\right)$$

$$= -\frac{4P}{\pi} \times .3354$$

Section 5.2.4 A well-established solution and comparison of results.

A solution to this problem has been given by C. A. M. Gray in reference 5 p. 266 .

In this paper a slightly different transformation is used but the equations are otherwise the same. The method of solution of the equations is less sophisticated and the results are, for P = .5 and width 2h,

$$\Omega'_z = \frac{2P}{\pi}\left(- .658 + 1.790\xi^2 - 2.050\xi^4 + 1.958\xi^6 - \ldots\right)$$

which is very close to the solution here.

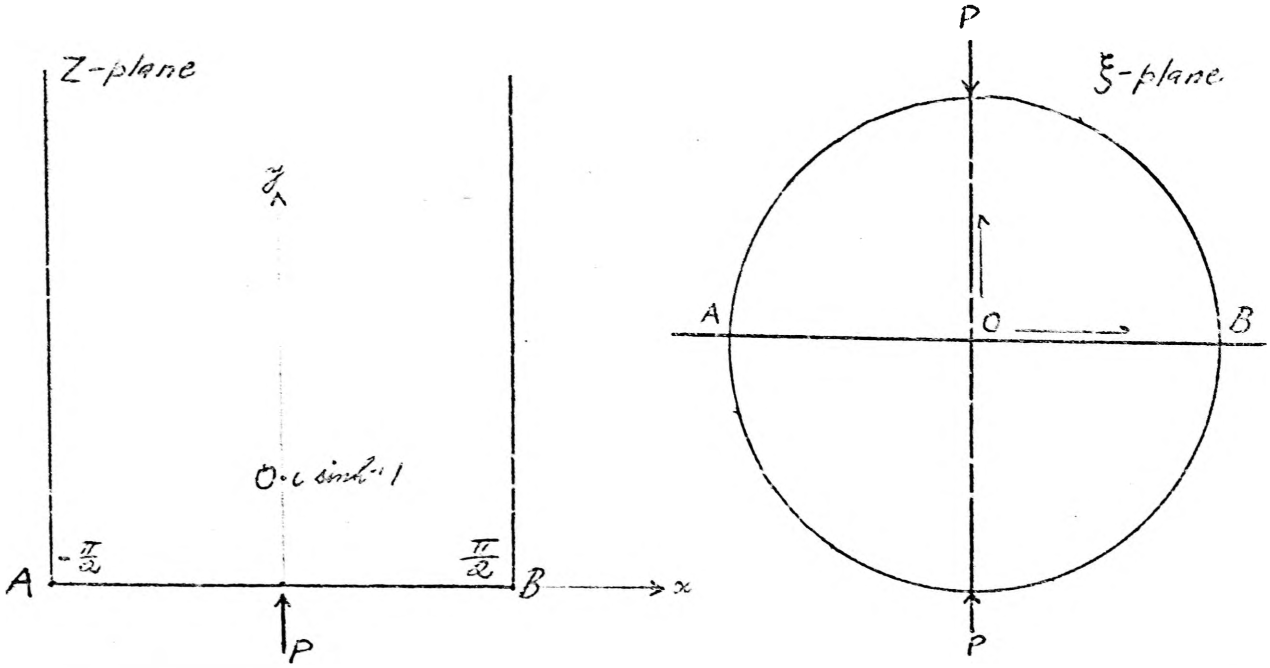
SECTION 6

CALCULATION OF STRESSES IN CANTILEVER WITH CONCENTRATED LOAD IN CENTRE OF END

The case considered here is one of compression only. A column of width π and infinite length is under plain strain from a normal point load applied in the centre of the end.

SECTION 6.1

TRANSFORMATION FUNCTION AND DERIVATION OF SIMULTANEOUS EQUATIONS.



$$z = \sin^{-1} \frac{i+\xi}{1+i\xi} \qquad \xi = \frac{\sin z - i}{1 - i \sin z}$$

$$\frac{dz}{d\xi} = \frac{1+i\xi}{\sqrt{1+2i\xi-\xi^2-2i\xi+1}} \cdot \frac{(1+i\xi)-(i+\xi)i}{(1+i\xi)^2}$$

$$= \frac{1+i\xi}{\sqrt{2}\sqrt{1-\xi^2}} \cdot \frac{\sqrt{2}}{(1+i\xi)^2}$$

$$= \frac{\sqrt{2}}{(1+i\xi)\sqrt{1-\xi^2}}$$

Now, expressing $1/(1+i\xi)$ and $1/(1-\xi^2)^{\frac{1}{2}}$ as infinite series, we obtain

$$\frac{dz}{d\xi} = \sqrt{2}\left(1 - i\xi - \xi^2 + i\xi^3 + \xi^4 - i\xi^5 - \xi^6 + \cdots\right)\left(1 + \tfrac{1}{2}\xi^2 + \tfrac{3}{8}\xi^4 + \tfrac{15}{48}\xi^6 + \tfrac{15.7}{48.8}\xi^8 + \tfrac{15.7.9}{48.8.10}\xi^{10} + \cdots\right)$$

$$= \sqrt{2}\left\{1 + \xi(-i) + \xi^2(\tfrac{1}{2}-1) - i\xi^3(\tfrac{1}{2}-1) + \xi^4(\tfrac{3}{8}-\tfrac{1}{2}+1) - i\xi^5(\tfrac{3}{8}-\tfrac{1}{2}+1) + \xi^6(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) - i\xi^7(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) + \ldots\right\}$$

The above values calculated for the U_r are substituted in equations (3.3) to (3.6) (and the other implied equations in that group) and the corresponding equations thus obtained are given the names E0, E1, E2 and so on. That is, eqn. (3.3) becomes eqn. E0, eqn. (3.4) becomes eqn. E1, and so on.

Integrating with respect to ξ gives :

$$z = u_0 + \sqrt{2}\Big[\xi - \frac{i\xi^2}{2} + \left(\tfrac{1}{2}-1\right)\frac{\xi^3}{3} - i\left(\tfrac{1}{2}-1\right)\frac{\xi^4}{4} + \left(\tfrac{3}{8}-\tfrac{1}{2}+1\right)\frac{\xi^5}{5}$$
$$- i\left(\tfrac{3}{8}-\tfrac{1}{2}+1\right)\frac{\xi^6}{6} + \left(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1\right)\frac{\xi^7}{7} - i\left(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1\right)\frac{\xi^8}{8} + \cdots\Big]$$

and $u_0 = z_{\xi=0} = \sin^{-1}1$

This gives us, with our usual terminology,

$u_1 = \sqrt{2}$ $\qquad$ $u_2 = -\frac{i}{2}\sqrt{2}$

$u_3 = (\frac{1}{2}-1)\frac{1}{3}\sqrt{2}$ $\qquad$ $u_4 = -i(\frac{1}{2}-1)\frac{1}{4}\sqrt{2}$

$u_5 = (\frac{3}{8}-\frac{1}{2}+1)\frac{1}{5}\sqrt{2}$ $\qquad$ $u_6 = -i(\frac{3}{8}-\frac{1}{2}+1)\frac{1}{6}\sqrt{2}$

. . . .

$$u_{2r+1} = \left(\sum_{s=0}^{r-1} \frac{1.3.5\ldots(2r-1-2s)}{2.4.6\ldots(2r-2s)}(-1)^s + (-1)^r\right)\frac{\sqrt{2}}{(2r+1)}$$

$$u_{2r+2} = -i\left(\sum_{s=0}^{r-1} \frac{1.3.5\ldots(2r-1-2s)}{2.4.6\ldots(2r-2s)}(-1)^s + (-1)^r\right)\frac{\sqrt{2}}{(2r+2)} = -i\,\frac{2r+1}{2r+2}u_{2r+1}$$

So eqns. 3.3 to 3.6 on p 14 become :

E0. b_0

$$+ \left(\overline{b}_0 - \frac{i}{2}\overline{b}_1 + \frac{1}{3}(\tfrac{1}{2}-1)\overline{b}_2 - \frac{i}{4}(\tfrac{1}{2}-1)\overline{b}_3 + \ldots\right) = \frac{P_0}{\sqrt{2}}$$

E1. $b_1 - ib_0$

$$+2\left(-\frac{i}{2}\overline{b}_0 + \frac{1}{3}(\tfrac{1}{2}-1)\overline{b}_1 - \frac{i}{4}(\tfrac{1}{2}-1)\overline{b}_2 + \frac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_3 - \ldots\right) = \frac{P_1}{\sqrt{2}}$$

E2. $b_2 - ib_1 + (\frac{1}{2}-1)b_0$

$$+3\left(\frac{i}{3}(\tfrac{1}{2}-1)\overline{b}_0 - \frac{i}{4}(\tfrac{1}{2}-1)\overline{b}_1 + \frac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_2 - \frac{i}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_3 \ldots\right) = \frac{P_2}{\sqrt{2}}$$

E3. $b_3 - ib_2 + (\frac{1}{2}-1)b_1 - i(\frac{1}{2}-1)b_0$

$$+4\left(-\frac{i}{4}(\tfrac{1}{2}-1)\overline{b}_0 + \frac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_1 - \frac{i}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_2 + \cdots\right] = \frac{P_3}{\sqrt{2}}$$

E4. $b_4 - ib_3 + (\frac{1}{2}-1)b_2 - i(\frac{1}{2}-1)b_1 + (\frac{3}{8}-\frac{1}{2}+1)b_0$

$$+5\left[\frac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_0 - \frac{i}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_1 + \frac{1}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\overline{b}_2 - \cdots\right] = \frac{P_4}{\sqrt{2}}$$

E5. $b_5 - ib_4 + (\frac{1}{2}-1)b_3 - i(\frac{1}{2}-1)b_2 + (\frac{3}{8}-\frac{1}{2}+1)b_1 - i(\frac{3}{8}-\frac{1}{2}+1)b_0$

$$+6\left[-\frac{i}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\overline{b}_0 + \frac{1}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\overline{b}_1 - \frac{i}{8}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\overline{b}_2 + \cdots\right] = \frac{P_5}{\sqrt{2}}$$

The equation obtained by adding eqn. (EI) and i times eqn. (E0) is given the name "(EQN Δ1)", the equation obtained by adding eqn. (E2) and i times eqn. (EI) is given the name "(EQN Δ2)" and so on.

E6. $$b_6 - ib_5 + (\tfrac{1}{2}-1)b_4 - i(\tfrac{1}{2}-1)b_3 + (\tfrac{3}{8}-\tfrac{1}{2}+1)b_2 - i(\tfrac{3}{8}-\tfrac{1}{2}+1)b_1 + (\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)b_0$$
$$+ 7\left[\tfrac{1}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\overline{b}_0 - \tfrac{i}{8}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\overline{b}_1 + \quad \cdots \quad\right] = \frac{P_6}{\sqrt{2}}$$

Now $P_0 = -\dfrac{4P}{\pi}$ $\qquad P_1 = 0$

$P_2 = +\dfrac{4P}{\pi}$ $\qquad P_3 = 0$

$P_4 = -\dfrac{4P}{\pi}$ $\qquad P_5 = 0$ $\qquad$ and so on.

So r.h.s's. of previous eqns. are :

$-\dfrac{4}{\sqrt{2}\pi} P$, $\qquad 0$

$+\dfrac{4}{\sqrt{2}\pi} P$, $\qquad 0$

$-\dfrac{4}{\sqrt{2}\pi} P$, $\qquad 0$ $\qquad$ and so on.

For the moment we take them as

-1, 0, +1, 0, -1, 0, +1, 0, etc. i.e. let $P = \dfrac{\sqrt{2}\pi}{4}$ and with this assumption, we calculate our b_0 , b_1 , b_2 , etc. and then multiply each by $\dfrac{4}{\sqrt{2}\pi}$.

SECTION 6.2

RE-ORGANIZATION OF EQUATIONS.

Add each equation to the previous multiplied by i (noting r.h s's. are ±1 or 0).

E1 + iE0 (EQN Δ1) $\qquad b_1 +$

$$\overline{b}_1\left(\tfrac{2}{3}(\tfrac{1}{2}-1) + \tfrac{1}{2}\right) + \overline{b}_2 i\left(-\tfrac{2}{4}(\tfrac{1}{2}-1) + \tfrac{1}{3}(\tfrac{1}{2}-1)\right) + \overline{b}_3\left(\tfrac{2}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right.$$
$$\left.+ \tfrac{1}{4}(\tfrac{1}{2}\ 1)\right) + \overline{b}_4 i\left(-\tfrac{2}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \tfrac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right) + \overline{b}_5\left(\tfrac{2}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\right.$$
$$\left.+ \tfrac{1}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right) + \ldots \qquad = -i$$

E2 + iE1 (EQN Δ2) $\qquad b_2 + \tfrac{1}{2}b_0 +$

$$\overline{b}_0\left(\tfrac{3}{3}(\tfrac{1}{2}-1) + 1\right) + \overline{b}_1 i\left(-\tfrac{3}{4}(\tfrac{1}{2}-1) + \tfrac{2}{3}(\tfrac{1}{2}-1)\right) + \overline{b}_2\left(\tfrac{3}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right.$$
$$\left.+ \tfrac{2}{4}(\tfrac{1}{2}-1)\right) + \overline{b}_3 i\left(-\tfrac{3}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \tfrac{2}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right)$$

$$+\ \overline{b_4}\left(\frac{3}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{2}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)+\ldots=1$$

<u>E3 + iE2</u> (EQN Δ3) $\quad b_3+\frac{1}{2}b_1+$

$$\overline{b_1}\ \left(\frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{4}\left(\frac{1}{2}-1\right)\right)+\overline{b_2}\,i\left(-\frac{4}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)$$

$$+\ \overline{b_3}\ \left(\frac{4}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{3}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)+\ \ldots\ =i$$

<u>E4 + iE3</u> (EQN Δ4) $\quad b_4+\frac{1}{2}b_2+\frac{3}{8}b_0$

$$+\ \overline{b_0}\ \left(\frac{5}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{4}{4}\left(\frac{1}{2}-1\right)\right)+\overline{b_1}\,i\left(-\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)$$

$$+\ \overline{b_2}\ \left(\frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{4}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)+\overline{b_3}\,i\left(-\frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right.$$

$$\left.+\frac{4}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\overline{b_4}\left(\frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{3}-\frac{1}{2}+1\right)+\frac{4}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\ldots=-1$$

<u>E5 + iE4</u> (EQN Δ5) $\quad b_5+\frac{1}{2}b_3+\frac{3}{8}b_1$

$$+\ \overline{b_1}\left(\frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)+i\overline{b_2}\left(-\frac{6}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$+\ \overline{b_3}\left(\frac{6}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$+i\,\overline{b_4}\left(-\frac{6}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right)+\ \ldots\ =-i$$

<u>E6 + iE5</u> (EQN Δ6) $\quad b_6+\frac{1}{2}b_4+\frac{3}{8}b_2+\frac{15}{48}b_0\qquad+\frac{15}{48}\overline{b_0}$

$$+\ \overline{b_0}\left(\frac{7}{7}\left(-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{6}{6}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\overline{b_1}\,i\left(-\frac{7}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right.$$

$$\left.+\frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\ \ldots$$

$$=1$$

Let $\quad b_n=\alpha_n+i\beta_n$

Take the real coefficients of equations Δ1, Δ3, Δ5, ...

$$\alpha_1\ +\alpha_1\ \left(\frac{2}{3}\left(\frac{1}{2}-1\right)+\frac{1}{2}\right)+\beta_2\ \left(-\frac{2}{4}\left(\frac{1}{2}-1\right)+\frac{1}{3}\left(\frac{1}{2}-1\right)\right)$$

$$+\ \alpha_3\ \left(\frac{2}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{1}{4}\left(\frac{1}{2}-1\right)\right)+\beta_4\ (\ .\ .\ .\ )+\ \ldots\ =0$$

$$\alpha_3+\frac{1}{2}\alpha_1+\alpha_1\ \left(\frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{4}\left(\frac{1}{2}-1\right)\right)+\beta_2(\ .\ .\ .\ )$$

$$+\ \alpha_3\ (\ .\ .\ .\ )+\ .\ .\ .\ =0\qquad .\quad .\quad .\quad .$$

and take the imaginary coefficients of equations

Δ2, Δ4, Δ6, ...

$$\beta_2 + \frac{1}{2}\beta_0 - \beta_0 \left(\frac{3}{3}(\tfrac{1}{2}-1) + 1\right) + \alpha_1 \left(-\frac{3}{4}(\tfrac{1}{2}-1) + \frac{2}{3}(\tfrac{1}{2}-1)\right)$$

$$-\beta_2 (\;.\quad.\quad.\;) + \alpha_3 (\;.\quad.\quad.\;) + \ldots = 0$$

$$\beta_4 + \frac{1}{2}\beta_2 + \frac{3}{8}\beta_0 - \beta_0 \frac{3}{8} + \alpha_1 (\;.\quad.\quad.\;) \ \ldots = 0$$

Obviously $\alpha_1 = \alpha_3 = \alpha_5 = \ldots = 0$

and $\beta_0 = \beta_2 = \beta_4 = \beta_6 = \ldots = 0$

is a solution of the above equations.

Now taking the real coefficients of equations E0, $\Delta 2$, $\Delta 4$, $\Delta 6$, ... and the imaginary coefficients of equations $\Delta 1$, $\Delta 3$, $\Delta 5$, ... (again simplifying the right hand side in the case of equation E0).

Real of E0	$\alpha_0 + (\alpha_0 - \frac{1}{2}\beta_1 + \frac{1}{3}(\frac{1}{2}-1)\,\alpha_2 - \frac{1}{4}(\frac{1}{2}-1)\,\beta_3 + \frac{1}{5}(\frac{3}{8}-\frac{1}{2}+1)\,\alpha_4 - \frac{1}{6}(\frac{3}{8}-\frac{1}{2}+1)\,\beta_5 \ldots = -1$	(6.0)
Imaginary of $\Delta 1$	$\beta_1 - \beta_1\left(\frac{2}{3}(\frac{1}{2}-1) + \frac{1}{2}\cdot 1\right) + \alpha_2\left(-\frac{2}{4}(\frac{1}{2}-1) + \frac{1}{3}(\frac{1}{2}-1)\right) - \beta_3\left(\frac{2}{5}(\frac{3}{8}-\frac{1}{2}+1) + \frac{1}{4}(\frac{1}{2}+1)\right) + \alpha_4\left(-\frac{2}{6}(\frac{3}{8}-\frac{1}{2}+1) + \frac{1}{5}(\frac{3}{8}-\frac{1}{2}+1)\right) - \beta_5\left(\frac{2}{7}(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1) + \frac{1}{6}(\frac{3}{8}-\frac{1}{2}+1)\right) + \;.\;.\;. = -1$	(6.1)
Real of $\Delta 2$	$\alpha_2 + \frac{1}{2}\alpha_0 + \alpha_0\left(\frac{3}{3}(\frac{1}{2}-1) + 1\right) + \beta_1\left(-\frac{3}{4}(\frac{1}{2}-1) + \frac{2}{3}(\frac{1}{2}-1)\right) + \alpha_2\left(\frac{3}{5}(\frac{3}{8}-\frac{1}{2}+1) + \frac{2}{4}(\frac{1}{2}-1)\right) + \beta_3\left(-\frac{3}{6}(\frac{3}{8}-\frac{1}{2}+1) + \frac{2}{5}(\frac{3}{8}-\frac{1}{2}+1)\right) + \alpha_4\left(\frac{3}{7}(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1) + \frac{2}{6}(\frac{3}{8}-\frac{1}{2}+1)\right) + \;.\;.\;. = 1$	(6.2)
Imaginary of $\Delta 3$	$\beta_3 + \frac{1}{2}\beta_1 - \beta_1\left(\frac{4}{5}(\frac{3}{8}-\frac{1}{2}+1) + \frac{3}{4}(\frac{1}{2}-1)\right) + \alpha_2\left(-\frac{4}{6}(\frac{3}{8}-\frac{1}{2}+1) + \frac{3}{5}(\frac{3}{8}-\frac{1}{2}+1)\right) - \beta_3\left(\frac{4}{7}(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1) + \frac{3}{6}(\frac{3}{8}-\frac{1}{2}+1)\right) + \alpha_4\left(-\frac{4}{8}(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1) + \frac{3}{7}(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1)\right) - \;.\;.\;.\;. = 1$	(6.3)

Real of $\Delta 4$

$$\alpha_4 + \frac{1}{2}\alpha_2 + \frac{3}{8}\alpha_0 \qquad + \frac{3}{8}\alpha_0 + \beta_1 \left(-\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right) + \frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right) + \alpha_2 \left(\frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right) + \frac{4}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)$$
$$+ \beta_3 \left(-\frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right) + \frac{4}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right) + \alpha_4 . \left(-\frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right) + \frac{4}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right) + \ldots = -1 \tag{6.4}$$

Imaginary of $\Delta 5$

$$\beta_5 + \frac{1}{2}\beta_3 + \frac{3}{8}\beta_1 \qquad - \beta_1 \left(\frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right) + -\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right) + \alpha_2 \left(-\frac{6}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right) + \frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$
$$- \beta_3 \left(\frac{6}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right) + \frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right) + \alpha_4 \left(-\frac{6}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{3}-\frac{1}{2}+1\right) + \frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right) \ldots = -1 \tag{6.5}$$

Real of $\Delta 6$

$$\alpha_6 + \frac{1}{2}\alpha_4 + \frac{3}{8}\alpha_2 + \frac{15}{48}\alpha_0 \qquad + \frac{15}{48}\alpha_0 + \beta_1 . \left(-\frac{7}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right) + \frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right) +$$
$$\alpha_2 \left(\frac{7}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right) + \frac{6}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right) + \beta_3 \left(-\frac{7}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right) + \frac{6}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right) + \ldots = 1 \tag{6.6}$$

SECTION 6.3

RESULTS OBTAINED FOR b_n BY ITERATION PROGRAMME

For the purpose of programming, we let

$\alpha_0 = x_1$, $\beta_1 = x_2$, $\alpha_2 = x_3$, $\beta_3 = x_4$, etc.

and solve for the x's.

ITERATION PROGRAMME

The equations were solved by iteration as previously. The programme looped 40 times, thus calculating x_1 to x_{40}. The value found for each x_r oscillated over a cycle of 4 iterations, each 4th giving approximately the same result. The average of each 4 iterations varied very little - see the table and the programme listing.

This average could be assumed to be the most likely correct value (this will be borne out later by the successive approximation method).

Note the little difference between the averages of

$$x_{29},\quad x_{33}$$
$$x_{30},\quad x_{34}$$
$$x_{31},\quad x_{35}$$
$$x_{32},\quad x_{36}$$

The right hand sides in this programme were actually taken to be ±100.

The values shown are 100 times the values obtained by taking r.h.s's = ±1.

Note that, after division by 100, the resultant values have to be multiplied by

$\frac{4P}{\pi\sqrt{2}}$ to give α_0, β_1, α_2, β_3, etc.

Also x_n, for high values of n is very nearly equal to $\pm\sqrt{2} = 1.4142$ so the answers are $\pm\frac{4P}{\pi}$.

See Appendix C Section C.1 for details of the programme CANT ITER.

	x1	x2	x3	x4
		last 4 iterations		
	-66.34	-120.37	145.00	140.18
	-72.21	-111.26	153.96	135.22
	-65.95	-119.32	146.08	142.73
	-72.65	-111.55	153.83	135.54
SUM	-277.15	-462.50	598.87	553.67
		previous 4 iterations		
	-66.20	-120.69	144.71	140.15
	-72.36	-111.03	154.24	134.95
	-65.73	-119.44	145.92	143.08
	-72.88	-111.26	154.05	135.48
SUM	-277.17	-462.42	598.92	553.66
AVGE	-69.29	-115.62	149.73	138.41

	x5	x6	x7	x8
		last 4 iterations		
	-141.92	-145.37	138.26	141.28
	-144.28	-140.43	142.28	138.54
	-140.58	-142.75	139.53	144.01
	-145.06	-139.33	141.36	139.94
SUM	-571.84	-567.88	561.43	563.77
		previous 4 iterations		
	-141.93	-145.62	138.14	141.18
	-144.30	-140.46	142.44	138.31
	-140.43	-142.65	139.54	144.23
	-145.25	-139.04	141.35	140.03
SUM	-571.91	-567.77	561.47	563.75
AVGE	-142.97	-141.95	140.36	140.94

x29		x30	x31	x32
		last 4 iterations		
	-141.61	-142.79	140.50	140.94
	-141.51	-141.99	142.22	139.88
	-140.78	-140.93	141.30	142.71
	-142.64	-139.80	141.12	142.06
SUM	-566.54	-565.51	565.14	565.59
AVGE	-141.63	-141.38	141.28	141.40
x33		x34	x35	x36
		last 4 iterations		
	-141.60	-142.66	140.61	140.95
	-141.45	-142.04	142.17	139.97
	-140.84	-140.94	141.35	142.60
	-142.51	-139.89	141.11	141.82
SUM	-566.40	-565.53	565.24	565.34
AVGE	-141.60	-141.38	141.31	141.34

SECTION 6.4

RESULTS OBTAINED FOR b_n BY SUCCESSIVE APPROXIMATION METHOD .

Section 6.4.1 Calculation of initial values (Method 1)

As a first approximation, let

$\beta_1 = -\sqrt{2}$ $\alpha_2 = \sqrt{2}$

$\beta_3 = \sqrt{2}$ $\alpha_4 = -\sqrt{2}$

$\beta_5 = -\sqrt{2}$ $\alpha_6 = \sqrt{2}$ and so on ,

and, for compatibility, let $\alpha_0 = -\sqrt{2}$.

This value was chosen because

$$\frac{1}{\sqrt{2}} = 1 - \tfrac{1}{2} + \tfrac{3}{8} - \tfrac{15}{48} + \tfrac{105}{384} - \ldots$$

which will make the first section of the equations (eqns. 6.0 to 6.6 shown) approach the right hand side for high-numbered equations. Also, the solutions found by iteration approach $\pm\sqrt{2}$. For the moment, allow $\beta_1 = -1$, $\alpha_2 = 1$, $\beta_3 = 1$, $\alpha_4 = -1$, etc. and calculate the r.h.s's.. This then must be multiplied by $\sqrt{2}$.

Then l.h.s. of eqn. (6.1) is

$$-1 + \left(\frac{2}{3}(\tfrac{1}{2}-1) + \frac{1}{2}\right) + \left(-\frac{2}{4}(\tfrac{1}{2}-1) + \frac{1}{3}(\tfrac{1}{2}-1)\right) - \left(\frac{2}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \frac{1}{4}(\tfrac{1}{2}-1)\right)$$
$$-\left(-\frac{2}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \frac{1}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right) + \left(\frac{2}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) + \frac{1}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right)$$
$$+\left(-\frac{2}{8}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) + \frac{1}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1)\right) - \ldots \qquad \text{EQN (1A)}$$

$$\doteq -1 + 1\left(-\tfrac{2}{3}+\tfrac{2}{4}-\tfrac{1}{3}-\tfrac{2}{5}+\tfrac{1}{4}+\tfrac{2}{6}-\tfrac{1}{5}-\ldots\right)$$
$$+\frac{1}{2}\left(\tfrac{2}{3}+1-\tfrac{2}{4}+\tfrac{1}{3}+\tfrac{2}{5}-\tfrac{1}{4}-\tfrac{2}{6}+\tfrac{1}{5}+\ldots\right)$$
$$+\frac{3}{8}\left(-2(\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\ldots) - (\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\ldots)\right)$$

$$\doteq -1 + 1\left(-2(\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\ldots) - (\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\ldots)\right) + \frac{1}{2}\left(1 + 2(\tfrac{1}{3}-\tfrac{1}{4}+\ldots)\right.$$
$$\left.+(\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\ldots)\right) + \frac{3}{8}\left(-2(\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\ldots) - (\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\ldots)\right)$$
$$+\frac{15}{48}\left(2(\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{9}-\ldots) + (\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{9}-\ldots)\right)$$
$$= -1 - 3(\log_e 2 - \overline{1-\tfrac{1}{2}}) + \frac{1}{2} + \frac{1}{2}.3(\log_e 2 - \overline{1-\tfrac{1}{2}})$$
$$-\frac{3}{8}.3(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}}) + \frac{15}{48}.3(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}} - \log_e 2) \ldots$$

(The above uses $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \log_e 2$)

l.h.s. of eqn. (6.2) gives

$$1 - \frac{1}{2} - \left((\tfrac{1}{2}-1) + 1\right) - \left(-\frac{3}{4}(\tfrac{1}{2}-1) + \frac{2}{3}(\tfrac{1}{2}-1)\right) + \left(\frac{3}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \frac{2}{4}(\tfrac{1}{2}-1)\right)$$
$$+\left(-\frac{3}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1) + \frac{2}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right) - \left(\frac{3}{7}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) + \frac{2}{6}(\tfrac{3}{8}-\tfrac{1}{2}+1)\right)$$
$$-\left(-\frac{3}{8}(\tfrac{15}{48}-\tfrac{3}{8}+\tfrac{1}{2}-1) + \frac{2}{7}(\tfrac{15}{48}-\tfrac{3}{3}+\tfrac{1}{2}-1)\right) + \ldots \qquad \text{EQN (2A)}$$

$$1-\frac{1}{2} \qquad + \left(\frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{4}\left(\frac{1}{2}-1\right)\right)+\left(-\frac{4}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)$$

$$-\left(+\frac{4}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{3}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)-\left(-\frac{4}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{3}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$+\left(\frac{4}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{3}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\ .\ .\ . \qquad \text{EQN (3A)}$$

$$=1-\frac{1}{2}+1\left(\frac{4}{5}-\frac{3}{4}-\frac{4}{6}+\frac{3}{5}+\frac{4}{7}\ldots\right)+\frac{1}{2}\left(-\frac{4}{5}+\frac{3}{4}+\frac{4}{6}-\frac{3}{5}-\frac{4}{7}+\frac{3}{6}\ldots\right)$$

$$+\frac{3}{8}\left(\frac{4}{5}-\frac{4}{6}+\frac{3}{5}+\frac{4}{7}-\frac{3}{6}\ldots\right)+\frac{15}{48}\left(-\frac{4}{7}+\frac{4}{8}-\frac{3}{7}-\frac{4}{9}+\frac{3}{8}\ldots\right)$$

$$+\quad .\quad .\quad .$$

$$=1-\frac{1}{2} \qquad -\frac{3}{4} \qquad +7\left(\log_e 2-\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}\right)+\frac{3}{8}$$

$$+\frac{1}{2}.7\left(\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}-\log_e 2\right)+\frac{3}{8}.7\left(\log_e 2-\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}\right)$$

$$+\frac{15}{48}.7\left(\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}}-\log_e 2\right)+\ .\ .\ .$$

l.h.s. of eqn.(6.4) gives

$$-1+\frac{1}{2}-\frac{3}{8} \qquad -\frac{3}{8} \qquad -\left(-\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)+\frac{4}{5}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)$$

$$+\left(\frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{4}{6}\left(\frac{3}{6}-\frac{1}{2}+1\right)\right)+\left(-\frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{4}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$-\left(\frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{4}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\left(-\frac{5}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right.$$

$$\left.+\frac{4}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right)-\ .\ \ .\ \ . \qquad \text{EQN (4A)}$$

$$\begin{aligned}
&= 0 + 1\left(-\tfrac{3}{4}+\tfrac{2}{3}+\tfrac{3}{5}-\tfrac{2}{4}-\tfrac{3}{6}+\tfrac{2}{5}\ \ldots\right)\\
&+ \frac{1}{2}\left(\tfrac{3}{4}-\tfrac{2}{3}-\tfrac{3}{5}+\tfrac{2}{4}+\tfrac{3}{6}-\tfrac{2}{5}\ \ldots\right)\\
&+ \frac{3}{8}\left(\tfrac{3}{5}-\tfrac{3}{6}+\tfrac{2}{5}+\tfrac{3}{7}-\tfrac{2}{6}\ \ldots\right)\\
&+ \frac{15}{48}\left(-\tfrac{3}{7}+\tfrac{3}{8}-\tfrac{2}{7}-\tfrac{3}{9}+\tfrac{2}{8}\ \ldots\right)\\
&+ \quad . \quad . \quad . \quad .\\
&= \frac{2}{3} + 1 \,.\, 5\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}}-\log_e 2\right)-\frac{2}{3}\,.\,\frac{1}{2}\\
&+ \frac{1}{2}\,.\, 5\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}}\right)\\
&+ \frac{3}{8}\,.\, 5\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}}\right)\\
&+ \frac{15}{48}\,.\, 5\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}}-\log_e 2\right) + \ldots
\end{aligned}$$

l.h.s. of eqn.(6.3) gives

$$= -1 + \frac{1}{2} - \frac{3}{8} \qquad -\frac{3}{8} \qquad +1\left(\frac{5}{6} - \frac{4}{5} - \frac{5}{7} + \frac{4}{6} + \frac{5}{8} - \frac{4}{7}\right)$$

$$+\frac{1}{2}\left(-\frac{5}{6} + \frac{4}{5} + \frac{5}{7} - \frac{4}{6} + \cdot \cdot \cdot\right)$$

$$+\frac{3}{8}\left(\frac{5}{6} - \frac{4}{5} - \frac{5}{7} + \frac{4}{6} \cdot \cdot \cdot\right)$$

$$+\frac{15}{48}\left(\frac{5}{7} - \frac{5}{8} + \frac{4}{7} + \frac{5}{9} - \frac{4}{8} \cdot \cdot \cdot\right)$$

$$+\frac{105}{384}\left(-\frac{5}{9} + \frac{5}{10} - \frac{4}{9} \cdot \cdot \cdot\right)$$

$$= -1 + \frac{1}{2} - \frac{3}{8} \qquad -\frac{3}{8} \qquad -\frac{4}{5} + 1.9\left(\overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}} - \log_e 2\right)$$

$$+\frac{1}{2} \cdot \frac{4}{5} \qquad +\frac{1}{2} \cdot 9\left(\log_e 2 - \overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}}\right)$$

$$-\frac{3}{8} \cdot \frac{4}{5} \qquad +\frac{3}{8} \cdot 9\left(\overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}} - \log_e 2\right)$$

$$+\frac{15}{48} \cdot 9\left(\log_e 2 - \overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}}\right)$$

$$+ \quad \cdot \quad \cdot \quad \cdot$$

l.h.s. of eqn.(6.5) gives

$$-1+\frac{1}{2}-\frac{3}{8}+\left(\frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{5}{6}\left(\frac{3}{8}-\frac{1}{2}+1\right)\right)+\left(-\frac{6}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right.$$

$$\left.+\frac{5}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)-\left(\frac{6}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{5}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$-\left(-\frac{6}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{5}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right)\dots \qquad \text{EQN (5A)}$$

$$=-1+\frac{1}{2}-\frac{3}{8}+1\left(-\frac{6}{7}+\frac{5}{6}+\frac{6}{8}\ \frac{5}{7}\ \frac{6}{9}+\frac{5}{8}\ .\ .\right)+\frac{1}{2}\left(\frac{6}{7}-\frac{5}{6}-\frac{6}{8}+\frac{5}{7}+\frac{6}{9}-\frac{5}{8}\dots\right)$$

$$+\frac{3}{8}\left(-\frac{6}{7}+\frac{5}{6}+\frac{6}{8}-\frac{5}{7}\dots\right)+\frac{15}{48}\left(\frac{6}{7}-\frac{6}{8}+\frac{5}{7}+\frac{6}{9}-\frac{5}{8}-\frac{6}{10}\dots\right)$$

$$+\frac{105}{384}\left(-\frac{6}{9}+\frac{6}{10}-\frac{5}{9}+\dots\right)\ .\ .\ .$$

$$=-1+\frac{1}{2}-\frac{3}{8}+\frac{5}{6}+1\,.11\left(\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}}-\log_e 2\right)-\frac{1}{2}\cdot\frac{5}{6}+\frac{1}{2}\,.11\left(\log_e 2\right.$$

$$\left.-\overline{1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}}\right)+\frac{3}{8}\cdot\frac{5}{6}+\frac{3}{8}\,.11\left(\overline{1-\frac{1}{2}+\dots-\frac{1}{6}}-\log_e 2\right)+\frac{15}{48}\,.11\left(\log_e 2\right.$$

$$\left.-\overline{1-\frac{1}{2}\dots-\frac{1}{6}}\right)+\frac{105}{384}\,.11\left(\overline{1-\frac{1}{2}\dots-\frac{1}{8}}-\log_e 2\right)+\ .\ .\ .$$

l.h.s. of eqn.(6.6) gives

$$1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48} \qquad -\frac{15}{48}-\left(-\frac{7}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)+\frac{6}{7}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)$$

$$+\left(\frac{7}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)+\frac{6}{8}\left(\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right)+\left(-\frac{7}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right.$$

$$\left.+\frac{6}{9}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right)-\left(\frac{7}{11}\left(\frac{945}{3840}-\frac{105}{384}+\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1\right)\right.$$

$$\left.+\frac{6}{10}\left(\frac{105}{384}-\frac{15}{48}+\frac{3}{8}-\frac{1}{2}+1\right)\right) \ldots \qquad \text{EQN (6A)}$$

$$=1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48} \qquad -\frac{15}{48}+1\left(-\frac{7}{8}+\frac{6}{7}+\frac{7}{9}-\frac{6}{8}-\frac{7}{10}+\frac{6}{9}\ldots\right)$$

$$+\frac{1}{2}\left(\frac{7}{8}-\frac{6}{7}-\frac{7}{9}+\frac{6}{8}+\frac{7}{10}-\frac{6}{9}\ldots\right)+\frac{3}{8}\left(-\frac{7}{8}+\frac{6}{7}+\frac{7}{9}-\frac{6}{8}-\frac{7}{10}+\frac{6}{9}\ldots\right)$$

$$+\frac{15}{48}\left(\frac{7}{8}-\frac{6}{7}-\frac{7}{9}+\frac{6}{8}+\ldots\right)+\frac{105}{384}\left(\frac{7}{9}-\frac{7}{10}+\frac{6}{9}+\frac{7}{11}-\frac{6}{10}\ldots\right)+\ldots$$

$$=1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48} \qquad -\frac{15}{48} \qquad +\frac{6}{7}\left(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}\right)$$

$$+1.13\left(\log_e 2-\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}}\right)+\frac{1}{2}.13\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots+\tfrac{1}{7}}-\log_e 2\right)$$

$$+\frac{3}{8}.13\left(\log_e 2-\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots+\tfrac{1}{7}}\right)+\frac{15}{48}.13\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots+\tfrac{1}{7}}-\log_e 2\right)$$

$$+\frac{105}{384}.13\left(\overline{\log_e 2-1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}}\right)+\ \ldots$$

The value of the left hand sides of equations (6.1) to (6.6) and the next two equations implied in this group are here grouped. The heading of each equation, "LHS (n)", refers to the value of the left hand side of eqn. (6.n).

Summarising these results and writing in a way which makes the trends obvious, gives :

<u>LHS (1)</u>

$$-1 \qquad\qquad + \frac{1}{2}.1$$

$$- 3\left(\log_e 2 - \overline{1-\tfrac{1}{2}}\right)\left(1-\tfrac{1}{2}\right) - 3\tfrac{3}{8}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}}\right)$$

$$+ 3.\tfrac{15}{48}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}}\right) - \dots$$

<u>LHS (3)</u>

$$\left(1-\tfrac{1}{2}\right) \qquad\qquad - \frac{3}{4}\left(1-\tfrac{1}{2}\right)$$

$$+ 7\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)$$

$$- 7\tfrac{15}{48}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}}\right)$$

$$= 7.\tfrac{105}{384}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\dots-\tfrac{1}{8}}\right) - \dots$$

<u>LHS (5)</u>

$$-\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right) \qquad\qquad + \frac{5}{6}\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)$$

$$- 11\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)$$

$$- 11.\tfrac{105}{384}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\dots-\tfrac{1}{8}}\right)$$

$$+ 11.\tfrac{945}{3840}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\dots-\tfrac{1}{10}}\right) - \dots$$

<u>LHS (7)</u>

$$\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right) \qquad\qquad - \frac{7}{8}\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)$$

$$+ 15\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\dots-\tfrac{1}{8}}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}+\tfrac{105}{384}\right)$$

$$- 15.\tfrac{945}{3840}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\dots-\tfrac{1}{10}}\right)$$

$$+ 15.\tfrac{10395}{46080}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\quad-\tfrac{1}{12}}\right) - \dots$$

$$\text{LHS (2)} \qquad \left(1-\tfrac{1}{2}\right) \qquad - \tfrac{1}{2} \qquad + \tfrac{2}{3}\left(1-\tfrac{1}{2}\right)$$

$$- 5\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}} - \log_e 2\right)\left(1-\tfrac{1}{2}\right)$$

$$+ 5.\tfrac{3}{8}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}}\right)$$

$$- 5.\tfrac{15}{48}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\ldots-\tfrac{1}{6}}\right) \qquad + \ldots$$

$$\text{LHS (4)} \qquad - \left(1-\tfrac{1}{2}+\tfrac{3}{8}\right) \qquad - \tfrac{3}{8} \qquad - \tfrac{4}{5}\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)$$

$$+ 9\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}} - \log_e 2\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)$$

$$+ 9.\tfrac{15}{48}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\ldots-\tfrac{1}{6}}\right)$$

$$- 9.\tfrac{105}{384}\left(\log_e 2 - \overline{1-\tfrac{1}{2}\ldots-\tfrac{1}{8}}\right) + \ldots$$

$$\text{LHS (6)} \qquad \left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right) \qquad - \tfrac{15}{48} \qquad + \tfrac{6}{7}\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)$$

$$- 13\left(\overline{1-\tfrac{1}{2}+\tfrac{1}{3}- \quad +\tfrac{1}{7}} - \log_e 2\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)$$

$$+ 13.\tfrac{105}{384}\left(\log_e 2 - \overline{1-\tfrac{1}{2}+\ldots-\tfrac{1}{8}}\right)$$

$$- 13.\tfrac{945}{3840}\left(\log_e 2 - \overline{1-\ldots-\tfrac{1}{10}}\right) + \ldots$$

$$\text{LHS (8)} \qquad - \left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}+\tfrac{105}{384}\right) - \tfrac{105}{384} \qquad - \tfrac{8}{9}\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}+\tfrac{105}{384}\right)$$

$$+ 17\left(\overline{1-\tfrac{1}{2}+\ldots+\tfrac{1}{9}} - \log_e 2\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}+\tfrac{105}{384}\right)$$

$$- 17.\tfrac{945}{3840}\left(\log_e 2 - \overline{1-\ldots-\tfrac{1}{10}}\right)$$

$$- 17.\tfrac{10395}{46080}\left(\log_e 2 - \overline{1-\ldots-\tfrac{1}{12}}\right) + \ldots$$

They have been split into the odd and even equations to show the patterns. In particular

1,5,9,...

3,7,11,...

2,6,10,...

4,8,12,...

go together .

A programme was written to evaluate these left hand sides and the following results obtained :

(1) -.86969	(2) .11614
(3) .69591	(4) -1.10900
(5) -.72218	(6) .36864
.	. .
(43) .70727	(44)-.82810
(45)-.70754	(46) .58907

Naturally only a certain number of terms were taken. The result oscillated depending on the number of terms taken and the mean of the oscillation was taken as the correct value.

These calculations are redone in programme CANTINF so this programme is not shown.In CANTINF, the value of the l.h.s. of equation 6.0 is also found.See the listing of CANTINF and also APPENDIX C Section C.3.

Note that the average for any 4 approaches $\frac{1}{\sqrt{2}}$. So, if we had taken $\pm\sqrt{2}$ as our original values the l.h.s's would approach ± 1, which indicates we are on the correct track in our initial value assumptions.

Section 6.4.2

Calculation of initial values (Method 2)

Before going on and successively approximating, we will calculate the l.h.s's by a second method and compare the results.

We make the same initial value assumptions

The expression headed "LHS (IA)" is again the value of the left hand side of eqn. (6.I). It is headed thus to show that it follows on from the expression labelled "EQN (IA)" on p.44. Similarly, the expression headed "LHS (2A)" is the value of the left hand side of eqn. (6.2), and follows on from the expression labelled "EQN (2A)". Others are headed similarly.

VIZ

$$\alpha_0 = \beta_1 = -1$$

$$\alpha_2 = \beta_3 = 1$$

$$\alpha_4 = \beta_5 = -1 \qquad \text{etc.}$$

<u>LHS(1A)</u>

$$-1+\frac{2}{3}\cdot\frac{1}{2}-\frac{2}{5}\cdot\frac{3}{8}+\frac{2}{7}\cdot\frac{15}{48}-\ .\ .\ .$$

$$+\left(\tfrac{1}{2}-\tfrac{2}{3}\right).1+\left(\tfrac{2}{4}-\tfrac{1}{3}\right)\left(1-\tfrac{1}{2}\right) \quad -\left(\tfrac{2}{5}-\tfrac{1}{4}\right)\left(1-\tfrac{1}{2}\right)$$

$$+\left(\tfrac{2}{6}-\tfrac{1}{5}\right)\left(\tfrac{3}{8}-\tfrac{1}{2}+1\right)-\left(\tfrac{2}{7}-\tfrac{1}{6}\right)\left(\tfrac{3}{8}-\tfrac{1}{2}+1\right)+\left(\tfrac{2}{8}-\tfrac{1}{7}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)$$

$$-\left(\tfrac{2}{9}-\tfrac{1}{8}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)+\left(\tfrac{2}{10}-\tfrac{1}{9}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}+\tfrac{105}{384}\right)-\ .\ .\ .$$

$$=-1+2\left(1-\sinh^{-1}1\right)-\frac{1}{1\cdot2\cdot3}+\left(1-\tfrac{1}{2}\right)\frac{1}{3\cdot4\cdot5}$$

$$+\left(\tfrac{3}{8}-\tfrac{1}{2}+1\right)\frac{3}{5\cdot6\cdot7}+\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)\frac{5}{7\cdot8\cdot9}-\ .\ .\ .$$

The above uses $\sinh^{-1}1=1-\frac{1}{2\cdot3}+\frac{3}{8\cdot5}-\frac{15}{48\cdot7}+\frac{105}{384\cdot9}\ .\ .\ .$

$$\left[\begin{array}{l} \sinh^{-1}x=\int(1+x^2)^{-\frac{1}{2}}dx \\ \qquad =\int\left(1-\tfrac{1}{2}x^2+\tfrac{3}{8}x^4-\tfrac{15}{48}x^6+\ .\ .\ .\right)dx \\ \qquad =x-\frac{1}{2\cdot3}x^3+\frac{3}{8\cdot5}x^5-\frac{15}{48\cdot7}x^7+\ .\ .\ . \\ \sinh^{-1}1=1-\frac{1}{2\cdot3}+\frac{3}{8\cdot5}-\frac{15}{48\cdot7}+\ .\ .\ . \end{array}\right]$$

<u>LHS (2A)</u>

$$\left(1-\tfrac{1}{2}\right)\ -\tfrac{3}{3}.\tfrac{1}{2}+\tfrac{3}{5}.\tfrac{3}{8}-\tfrac{3}{7}.\tfrac{15}{48}+\ .\ .\ .$$

$$+0 \quad -\left(\tfrac{3}{4}-\tfrac{2}{3}\right)\left(1-\tfrac{1}{2}\right)+\left(\tfrac{3}{5}-\tfrac{2}{4}\right)\left(1-\tfrac{1}{2}\right)-\left(\tfrac{3}{6}-\tfrac{2}{5}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)$$

$$+\left(\tfrac{3}{7}-\tfrac{2}{6}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)-\left(\tfrac{3}{8}-\tfrac{2}{7}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)+\left(\tfrac{3}{9}-\tfrac{2}{8}\right)\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)\ .\ .\ .$$

$$=\left(1-\tfrac{1}{2}\right)-3\left(1-\sinh^{-1}1\right)+\left(1-\tfrac{1}{2}\right)\frac{1}{3\cdot4\cdot5}-\left(1-\tfrac{1}{2}+\tfrac{3}{8}\right)\frac{1}{5\cdot6\cdot7}$$

$$-\left(1-\tfrac{1}{2}+\tfrac{3}{8}-\tfrac{15}{48}\right)\frac{3}{7\cdot8\cdot9}+\ .\ .\ .$$

LHS (3A) $(1-\frac{1}{2})+\frac{4}{5}\cdot\frac{3}{8}-\frac{4}{7}\cdot\frac{15}{48}+\ .\ .\ .$

$+(\frac{4}{5}-\frac{3}{4})(1-\frac{1}{2})\quad-(\frac{4}{6}-\frac{3}{5})(1-\frac{1}{2}+\frac{3}{8})+(\frac{4}{7}-\frac{3}{6})(1-\frac{1}{2}+\frac{3}{8})$

$-(\frac{4}{8}-\frac{3}{7})(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})+(\frac{4}{9}-\frac{3}{8})(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})$

$-(\frac{4}{10}-\frac{3}{9})(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}+\frac{105}{384})+(\frac{4}{11}-\frac{3}{10})(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}+\frac{105}{384})\ .\ .\ .$

$=(1-\frac{1}{2})+4\left(\sinh^{-1}1-(1-\frac{1}{2\cdot 3})\right)$

$+(1-\frac{1}{2})\frac{3}{3\cdot 4\cdot 5}+(1-\frac{1}{2}+\frac{3}{8})\frac{1}{5\cdot 6\cdot 7}-(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})\frac{1}{7\cdot 8\cdot 9}$

$-(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}+\frac{105}{384})\frac{3}{9\cdot 10\cdot 11}+\ .\ .\ .$

LHS (4A) $-(1-\frac{1}{2}+\frac{3}{8})\ -\frac{3}{8}\cdot\frac{5}{5}+\frac{5}{7}\cdot\frac{15}{48}-\frac{5}{9}\cdot\frac{105}{384}+\ .\ .\ .$

$+(\frac{5}{6}-\frac{4}{5})(1-\frac{1}{2}+\frac{3}{8})-(\frac{5}{7}-\frac{4}{6})(1-\frac{1}{2}+\frac{3}{8})+\left((\frac{5}{8}-\frac{4}{7})-(\frac{5}{9}-\frac{4}{8})\right)*$

$\left(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}\right)+\left((\frac{5}{10}-\frac{4}{9})-(\frac{5}{11}-\frac{4}{10})\right)(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}+\frac{105}{384})\ .\ .\ .$

$=-(1-\frac{1}{2}+\frac{3}{8})\ -5\left(\sinh^{-1}1-(1-\frac{1}{2\cdot 3})\right)$

$-(1-\frac{1}{2}+\frac{3}{8})\cdot\frac{3}{5\cdot 6\cdot 7}-(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})\cdot\frac{1}{7\cdot 8\cdot 9}$

$+(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48}+\frac{105}{384})\cdot\frac{1}{9\cdot 10\cdot 11}+\ .\ .\ .$

LHS (5A) $-(1-\frac{1}{2}+\frac{3}{8})+\frac{6}{7}\cdot\frac{15}{48}-\frac{6}{9}\cdot\frac{105}{384}+\ .\ .\ .$

$-(\frac{6}{7}-\frac{5}{6})(1-\frac{1}{2}+\frac{3}{8})+\left((\frac{6}{8}-\frac{5}{7})-(\frac{6}{9}-\frac{5}{8})\right)(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})$

$-\left((\frac{6}{10}-\frac{5}{9})-(\frac{6}{11}-\frac{5}{10})\right)(1-\ .\ .\ .+\frac{105}{384})$

$=-(1-\frac{1}{2}+\frac{3}{8})+6\left((1-\frac{1}{2\cdot 3}+\frac{3}{8\cdot 5})-\sinh^{-1}1\right)$

$-(1-\frac{1}{2}+\frac{3}{8})\frac{5}{5\cdot 6\cdot 7}-(1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48})\cdot\frac{3}{7\cdot 8\cdot 9}$

$+(1-\frac{1}{2}+\frac{3}{3}-\frac{15}{48})\cdot\frac{1}{9\cdot 10\cdot 11}+\ .\ .\ .$

etc.

Obviously these series converge very quickly. A programme was written to evaluate these left hand sides and the following results obtained:

(1) -.86948 (2) .11594

(3) .69610 (4) -1.10918

(5) -.72199 (6) .36854

These results were very close to those obtained by the first method. For further details see APPENDIX Section C.2 and computer programmes RESIDUAL 2 RUN 1 and RESIDUAL 2 RUN2.

Section 6.4.3

RESULTS OBTAINED FOR b_n FROM PROGRAMME

The results of the first method were used to successively approximate.

A programme was written to do this, calculating the amounts which need to be added to the initial guesses to give the actual values.

This programme, CANTINF, is shown, and described in APPENDIX C Section C.3.

The results converged very rapidly and were:-

.72509, .25672, .07965, -0.02965, . . .

(See last 5 lines of listing output by CANTINF).

To obtain x_1, x_2, etc, these values must be added to $\pm\sqrt{2}$ i.e. 1.4 142

This then gives $x_1 = -.68912$

$x_2 = -1.15749$

$x_3 = 1.49386$

$x_4 = 1.38456$ etc.

Compare these with results obtained from the first programme which used straight iteration. They are very close to the average values calculated there and fall very nicely within the oscillation. Again note they must be multiplied by $\frac{4P}{\pi\sqrt{2}}$.

These values then are used in a programme to calculate the stresses.

This programme (CAN2) reads the values punched out on cards by the first programme and calculates and prints out the actual values of α_0, β_1, α_2, β_3, . . . and thus b_0, b_1, b_2, etc.

These results are:-

$b_0 = -.62042$

$b_1 = -1.04211i$

$b_2 = 1.34495$

$b_3 = 1.24655i$

$b_4 = 1.28698$

$b_5 = -1.27907i$

etc.

See programme listing for further details.

SECTION 6.5

CALCULATION OF STRESSES

Section 6.5.1 Theoretical derivation of $n\alpha_n$.

Before proceeding to calculate the stresses at various points by programme some more theory is required.

Remember $Z=u_0+u_1\xi+u_2\xi^2+\ .\ .\ .$

where $u_0=i\sinh^{-1}1$

$u_1=\sqrt{2}$

$u_2=-\frac{i}{2}\sqrt{2}$

$u_3=(\frac{1}{2}-1)\frac{1}{3}\sqrt{2}$

$u_4=-i(\frac{1}{2}-1)\frac{1}{4}\sqrt{2}$

$u_5=(\frac{3}{8}-\frac{1}{2}+1)\frac{1}{5}\sqrt{2}$

$u_6=-1(\frac{3}{8}-\frac{1}{2}+1)\frac{1}{6}\sqrt{2}$

etc.

and

EQN 3.8 $\quad a_1+\bar{u}_0b_1+\bar{u}_1b_2+\ .\ .\ .) = Q_0$

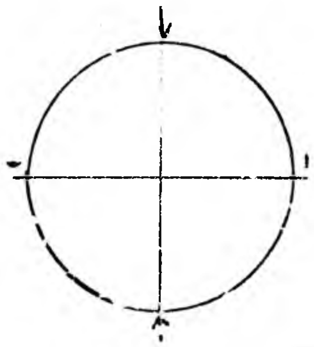
EQN 3.9 $\quad 2a_2+2(\bar{u}_0b_2+\bar{u}_1b_3+\ .\ .\ .) = Q_1$

EQN 3.10 $\quad 3a_3+3(\bar{u}_0b_3+\bar{u}_1b_4+\ .\ .\ .) = Q_2$

. . .

and for our stress system in the ξ plane,

VIZ



$Q_r = P_r$

Obviously, knowing Q_r's and b_r's, we can calculate the a_r's. However, the equations are not in a form to give an accurate calculation with a finite number of terms.

The values of b_n have been listed out in the programme CAN2 and are called x_n where

$$\left\{\begin{array}{ll} b_n = x_{n+1} & n \text{ even} \\ b_n = ix_{n+1} & n \text{ odd} \end{array}\right\}$$

Equation n referred to here is the nth equation in the group 3.8 to 3.10 (and other implied equations) shown on p.55. That is, it is the equation

$$na_n + n(\bar{u}_0 b_n + \bar{u}_1 b_{n+1} + \cdots) = Q_n$$

Also calculated and printed out are y_n, where

$$\left\{\begin{array}{ll} x_n = -\frac{4}{\pi} + y_n & n = 1,2,5,6,9,10,\ \dots \\ x_n = \frac{4}{\pi} + y_n & n = 3,4,7,8,11,12,\ \dots \end{array}\right\}$$

(the constant force P is ignored here and introduced when necessary.)

The y_n are of course merely $\frac{4}{\pi\sqrt{2}}$ times the values calculated by the first programme VIZ the amounts which have to be added to the initial guesses to give the actual values.

The results were

$y_1 = .62581$

$y_2 = .23113$

$y_3 = .07171$

$y_4 = -.02669$

$y_5 = -.01374$

$y_6 = -.00583$

$y_7 = -.01036$

$y_8 = -.00403$

$y_9 = -.00883$

$y_{10} = -.00222$

l.h.s. of EQN n becomes $\qquad n=1,5,9,13,\ \dots$

$$na_n + n\Big((i\bar{u}_0(-\tfrac{4}{\pi}+y_{n+1}) + \bar{u}_1(\tfrac{4}{\pi}+y_{n+2}) + i\bar{u}_2(\tfrac{4}{\pi}+y_{n+3})$$
$$+\bar{u}_3(-\tfrac{4}{\pi}+y_{n+4}) + i\bar{u}_4(-\tfrac{4}{\pi}+y_{n+5}) + \bar{u}_5(+\tfrac{4}{\pi}+y_{n+6}) + \dots\Big)$$
$$= na_n + n\left(i\bar{u}_0 y_{n+1} + \bar{u}_1 y_{n+2} + i\bar{u}_2 y_{n+3} + \dots\right)$$
$$+\frac{4n}{\pi}\left(-i\bar{u}_0 + \bar{u}_1 + i\bar{u}_2 - \bar{u}_3 - i\bar{u}_4 + \dots\right)$$

When $n=2,6,10,14,\ .\ .\ .$

l.h.s. of EQN n

$$=na_n+n\left(\overline{u}_0\left(\frac{4}{\pi}+y_{n+1}\right)+i\overline{u}_1\left(\frac{4}{\pi}+y_{n+2}\right)+\overline{u}_2\left(-\frac{4}{\pi}+y_{n+3}\right)+\right.$$
$$\left.+i\overline{u}_3\left(-\frac{4}{\pi}+y_{n+4}\right)+\overline{u}_4\left(\frac{4}{\pi}+y_{n+5}\right)+\ .\ .\ .\right)$$
$$=na_n+n\left(\overline{u}_0 y_{n+1}+i\overline{u}_1 y_{n+2}+\ .\ .\ .\right)+\frac{4n}{\pi}\left(\overline{u}_0+i\overline{u}_1-\overline{u}_2-i\overline{u}_3+\ldots\right)$$

Similarly when $n=3,7,11,15,\ .\ .\ .$

l.h.s. of EQN n

$$=na_n+n\left(\overline{u}_0 y_{n+1}+i\overline{u}_1 y_{n+2}+\ .\ .\ .\right)$$
$$+\frac{4n}{\pi}\left(i\overline{u}_0-\overline{u}_1-i\overline{u}_2+\overline{u}_3\ .\ .\ .\ \right)$$

and when $n=4,8,12,16,\ .\ .\ .$

l.h.s. of EQN n

$$=na_n+n\left(\overline{u}_0 y_{n+1}+\ .\ .\ .\right)+\frac{4n}{\pi}\left(-\overline{u}_0-i\overline{u}_1+\overline{u}_2+i\overline{u}_3\ .\ .\ .\right)$$

Note that the second infinite series is always

$-(\overline{u}_0-\overline{u}_2+\overline{u}_4-\ .\ .\ .)-i(\overline{u}_1-\overline{u}_3+\overline{u}_5-\ .\ .\ .)$ multiplied

by $\pm i$ or ± 1.

It is easy to show that this is zero:

$$Z=u_0+u_1\xi+u_2\xi^2+\ .\ .\ .$$

When $\xi=-i$ $\quad Z=0$ (see transform diagram)

and $Z_{(\xi=-i)}=u_0-iu_1-u_2+iu_3+u_4-\ .\ .\ .$

$$=u_0-u_2+u_4-u_6\ .\ .\ .$$
$$-i(u_1-u_3+u_5-\ .\ .\ .) \qquad \text{(EQN 6.11)}$$

Since $u_0-u_2+u_4-u_6+\ .\ .\ .$ is purely imaginary

and $u_1-u_3+u_5$ is purely real,

$$Z_{(\xi=-i)}=-(\overline{u}_0-\overline{u}_2+\overline{u}_4-\ .\ .\ .)$$
$$-i(\overline{u}_1-\overline{u}_3+\overline{u}_5\ .\ .\ .)=0.$$

(Note that EQN 6.11 also proves

$$\sinh^{-1}1=\sqrt{2}\ \{1-\tfrac{1}{2}+\tfrac{1}{3}(1-\tfrac{1}{2})-\tfrac{1}{4}(1-\tfrac{1}{2})+\tfrac{1}{5}(1-\tfrac{1}{2}+\tfrac{3}{8})-\ .\ .\ .\})$$

So, for all equations,

$$\therefore\ na_n+n(\bar{u}_0y_{n+1}+\bar{u}_1y_{n+2}+\ .\ .\ .)P = Q_{n-1}$$

where u_r and y_n are complex.

Note that, since $y_n \to 0$ as $n\to\infty$ $a_n \to 0$ as $n\to\infty$

This does not prove, however that $na_n \to Q_{n-1}$ as $n\to\infty$, although empirically this seems so.

na_n is calculated from the programme results by using the equations just derived.

See APPENDIX C Section C.4 and programme CAN2, especially for the accuracy obtained in using the above series.

Section 6.5.2 Theoretical derivation of Ω', Ω'', ω'' and stresses.

Knowing b_n and na_n, we can now calculate Ω'_z, Ω''_z, and ω''_z and hence $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$ for any point z.

Find ξ, using $\xi = \dfrac{\sin z - i}{1 - i \sin z}$

Then use

$$\Omega'_z = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \ldots$$

$$\Omega''_z = (b_1 + 2b_2\xi^2 + 3b_3\xi^3 + 4b_4\xi^4 + \ldots)\frac{d\xi}{dz}$$

$$\omega''_z = (a_1 + 2a_2\xi + 3a_3\xi^2 + 4a_4\xi^3 + \quad)\frac{d\xi}{dz}$$

and $\dfrac{d\xi}{dz} = \dfrac{(1 + i\xi)\sqrt{1 - \xi^2}}{\sqrt{2}}$

As $b_n \to \pm\frac{4}{\pi}$ as $n \to \infty$ and na_n appears to approximate to $\pm\frac{4}{\pi}$ when n is large, the above series do not converge unless $\xi^n \to 0$.

Away from the boundaries, $\xi^n \to 0$ as $n \to \infty$ although not always quickly. On the boundaries, however, the magnitude of ξ is 1. To obtain values on the boundary, therefore, it is necessary to split the series into 2 parts, an analytically calculable one and a quickly convergent series which can be summed successfully by numerical means.

This is done as follows :-

$$\Omega'_z = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \ldots$$

$$= (-\frac{4}{\pi} + y_1) + (-\frac{4}{\pi} + y_2)i\xi + (\frac{4}{\pi} + y_3)\xi^2$$

$$+ (\frac{4}{\pi} + y_4)i\xi^3 + (-\frac{4}{\pi} + y_5)\xi^4 + (-\frac{4}{\pi} + y_6)i\xi^5 + \ldots$$

where the y_n have been found previously.

i.e. $$\Omega'_z = -\frac{4}{\pi} - \frac{4}{\pi}i\xi + \frac{4}{\pi}\xi^2 + \frac{4}{\pi}i\xi^3 - \frac{4}{\pi}\xi^4 - \frac{4}{\pi}i\xi^5 + \ldots$$

$$+ y_1 + y_2 i\xi + y_3\xi^2 + y_4 i\xi^3 + y_5\xi^4 + \ldots)$$

$$= -\frac{4}{\pi}(1 + i\xi - \xi^2 - i\xi^3 + \xi^4 + i\xi^5 - \ldots)$$

$$+ (y_1 + y_2 i\xi + y_3\xi^2 + y_4 i\xi^3 + y_5\xi^4 + \ldots)$$

$$\Omega'_z = -\frac{4}{\pi}\frac{1}{1 - i\xi} + (y_1 + y_2 i\xi + y_3\xi^2 + y_4 i\xi^3 + y_5\xi^4 + \ldots)$$

Similarly, or by differentiation of Ω'_z, we can find Ω''_z

e.g. $$\Omega''_z = \frac{\partial}{\partial\xi}\Omega'_z\frac{\partial\xi}{\partial z}$$

$$= \left(-\frac{4i}{\pi}\frac{1}{(1 - i\xi)^2} + (y_2 i + 2y_3\xi + 3y_4 i\xi^2 + 4y_5\xi^3 + 5y_6 i\xi^4 + \ldots)\right)\frac{(1+i\xi)\sqrt{1-\xi^2}}{\sqrt{2}}$$

$$\omega''_z = \left(a_1 + 2a_2\xi + 3a_3\xi^2 + 4a_4\xi^3 + \ldots\right)\frac{(1+i\xi)\sqrt{1-\xi^2}}{\sqrt{2}}$$

$$= \left((-\frac{4}{\pi} + \frac{4}{\pi}\xi^2 - \frac{4}{\pi}\xi^4 + \ldots) + (S_1 + S_2\xi + S_3\xi^2 + S_4\xi^3 + \ldots)\right)\frac{(1+i\xi)\sqrt{1-\xi^2}}{\sqrt{2}}$$

where $a_1 = -\frac{4}{\pi} + S_1$ i.e. $S_1 = -1.(\bar{u}_0 y_2 + \bar{u}_1 y_3 + \ldots)$

$2a_2 = 0 + S_2$ i.e. $S_2 = -2.(\bar{u}_0 y_2 + \bar{u}_1 y_3 + \ldots)$

$3a_3 = \frac{4}{\pi} + S_3$ and so on

and, in the programme CAN2, $S_n = AR + i.AI$

i.e. $\underline{\underline{\omega''_z = \left(-\frac{4}{\pi}\cdot\frac{1}{1+\xi^2} + S_1 + S_2\xi + S_3\xi^2 + \ldots\right)}}\frac{(1+i\xi)\sqrt{1-\xi^2}}{\sqrt{2}}$

Section 6.5.3 Computer calculation of stresses.

Some important points on the diagram can be calculated manually e.g.

$$\xi = -i \qquad \Omega'_z = -\infty \qquad \Omega''_z = -i\infty \qquad \omega''_z = -\infty$$

$$\xi = i \qquad \Omega'_z = -\frac{2}{\pi} + (y_1 - y_2 - y_3 + y_4 + y_5 \ldots)$$
$$\simeq -\frac{1}{\pi}$$

The programme gave it as -.31357 (about 1.5% Out)

$\Omega''_z = 0$ Programme gave -.00006

$\omega''_z = -\frac{2}{\pi}$. (This was actually made equal to $-\frac{2}{\pi}$ since $\omega''_z \to -\frac{2}{\pi}$ as $\xi \to i$ but if calculated in the programme by the normal way would give $-\frac{0}{0}$.)

The calculation of Ω''_z and ω''_z are still not particularly accurate on the boundaries since ny_n in Ω''_z and S_n in ω''_z are not negligibly small. The inaccuracies caused can be quite large, especially when $\xi \simeq 1$ since the ξ^n may change very slowly with n. The following diagram shows the stresses at various points on the shape. Each value must be multiplied by P, the intensity of the point load.

above here
$\xi \approx i$

Y	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$\frac{5\pi}{3}$	.00226 .31579 0	.00227 -.31580 .00004	.00228 -.31582 .00007	.00230 -.31585 .00010	.00233 -.31588 .00012	.00236 -.31592 .00013	.00240 -.31596 .00013
$\frac{4\pi}{3}$	.00622 .31546 0	.00597 -.31536 -.00067	.00529 -.31515 -.00112	.00438 -.31509 -.00124	.00351 -.31554 -.00111	.00295 -.31682 -.00119	.00255 -.31779 -.00253
$\frac{2\pi}{3}$	.07228 .38657 0	.06752 -.37749 -.02711	.05480 -.35274 -.04724	.03788 -.31855 -.05565	.02094 -.28267 -.05446	.00782 -.25293 -.03173	-.00801 -.22736 -.00128
$\frac{\pi}{3}$	.12903 -.64844 0	.09618 -.57964 -.12201	.03490 -.42989 -.16982	-.00434 -.28670 -.15081	-.01267 -.17892 -.10537	.00484 -.09229 -.05616	-.00555 .00748 .01138
$\frac{\pi}{6}$	.14580 -1.22968 0	-.04983 -.79246 -.37584	-.16228 -.31928 -.27758	-.12158 -.13029 -.13383	-.06195 -.05806 -.04870	-.01844 -.01052 -.00674	.00033 .06395 .00988
$\frac{\pi}{12}$	.15992 -2.43408 0	-.45135 -.61070 -.59899	-.24245 -.10092 -.17629	-.08825 -.02869 -.04514	-.02468 -.01135 .00213	-.00303 .00261 .01781	.00368 .01973 .00381
(0,0)		.19390 -.00968 -.01425	.18030 -.01366 .00040	.13364 -.00023 .01217	.07570 .00906 .00313	.02010 .00448 .00148	.00432 .00432 0

X

$P = 1$

$$\begin{bmatrix} \hat{x}x \\ \hat{y}y \\ \hat{x}y \end{bmatrix}$$

SECTION 6.5.4

COMPARISON OF STRESSES WITH THOSE CALCULATED PREVIOUSLY BY OTHER METHODS.

Refer to equation in middle of page 52 of reference 1

With $y=c-\frac{4}{3}\ell$ which is equivalent to $y=\frac{2\pi}{3}$ on our diagram,

$$\sigma_y=-\frac{P}{2\ell}-\frac{P}{\ell}\left(\frac{\frac{4}{3}\pi+1}{e^{\frac{4}{3}\pi}}\cos\frac{\pi x}{\ell}+\frac{\frac{8}{3}\pi+1}{e^{\frac{8}{3}\pi}}\cos\frac{2\pi x}{\ell}+\frac{4\pi+1}{e^{4\pi}}\cos\frac{3\pi x}{\ell}+\ldots\right)$$

$$=-\frac{P}{2\ell}\left(1+2\left(.0787\cos\frac{\pi x}{\ell}+.0022\cos\frac{2\pi x}{\ell}+.00004\cos\frac{3\pi x}{\ell}\ldots\right)\right)$$

In our case $\ell=\frac{\pi}{2}$,

$x=0$

$$\sigma_y=-\frac{P}{\pi}(1+2(.0809))=-\frac{P}{\pi}(1.1618)$$

$=-P\ (.370)$ cf. $-.387$ by this method

$x=\frac{\ell}{3}$

$$\sigma_y=-\frac{P}{\pi}\left(1+2\left(.0787\cos\frac{\pi}{3}+.0022\cos\frac{2\pi}{3}\right)\right)$$

$=-P(.343)$ cf. $-.352$ by this method

$x=\frac{2\ell}{3}$

$$\sigma_y=-\frac{P}{\pi}(1+2(.0787(-.5)+.0022(-.5)))$$

$=-P(.292)$ cf. $-.282$ by this method

With $y=c-\frac{2}{3}\ell$ i.e. $y=\frac{1}{3}\pi$ on our diagram

$$\sigma y=\frac{P}{2\ell}-\frac{P}{\ell}\left(\frac{\frac{2}{3}\pi+1}{e^{\frac{2}{3}\pi}}\cos\frac{\pi x}{\ell}+\frac{\frac{4}{3}\pi+1}{e^{\frac{4}{3}\pi}}\cos\frac{2\pi x}{\ell}+\frac{2\pi+1}{e^{2\pi}}\cos\frac{3\pi x}{\ell}+\frac{\frac{8}{3}\pi+1}{e^{\frac{8}{3}\pi}}\cos\frac{4\pi x}{\ell}\quad.\quad.\quad.\right)$$

$$=-\frac{P}{2\ell}\left(1+2\left(.3808\cos\frac{\pi x}{\ell}+.0778\cos\frac{2\pi x}{\ell}+.0136\cos\frac{3\pi x}{\ell}+.0022\cos\frac{4\pi x}{\ell}+.0003\cos\frac{5\pi x}{\ell}\quad.\quad.\quad.\right)\right)$$

Again $\ell=\frac{\pi}{2}$,

and $x=0$

$$\sigma y=-\frac{P}{\pi}\left(1+2(.4747)\right)=-P\left(\frac{1.949}{3.1416}\right)$$

$$\underline{\underline{=-P(.620)}}$$ cf. $\underline{\underline{-.649}}$ by this method

$x=\frac{\ell}{3}$

$$\sigma y=-\frac{P}{\pi}\left(1+2\{.3808(.5)+.0778(-.5)+.0136(-1)+.0022(-.5)+.0003(.5)\}\quad.\quad.\quad.\right)$$

$$\underline{\underline{=-P(.405)}}$$ cf. $\underline{\underline{-.430\text{ by this method}}}$

It is also useful to compare results close to the point of application of the load with corresponding points on the half plane.

		CANTILEVER	HALF-PLANE
$x=\frac{\pi}{12}$	$\widehat{xx}$	-.45135	-.608
$y=\frac{\pi}{12}$	$\widehat{yy}$	-.61070	-.608
	$\widehat{xy}$	-.59899	-.608
$x=\frac{\pi}{6}$	$\widehat{xx}$	-.24245	-.3891
$y=\frac{\pi}{12}$	$\widehat{yy}$	-.10092	-.0973
	$\widehat{xy}$	-.17629	-.1946

SECTION 7

INVESTIGATION OF STRESS DISTRIBUTION AT CHANGES OF SECTION OF MEMBERS UNDER TWO-DIMENSIONAL STRESS.

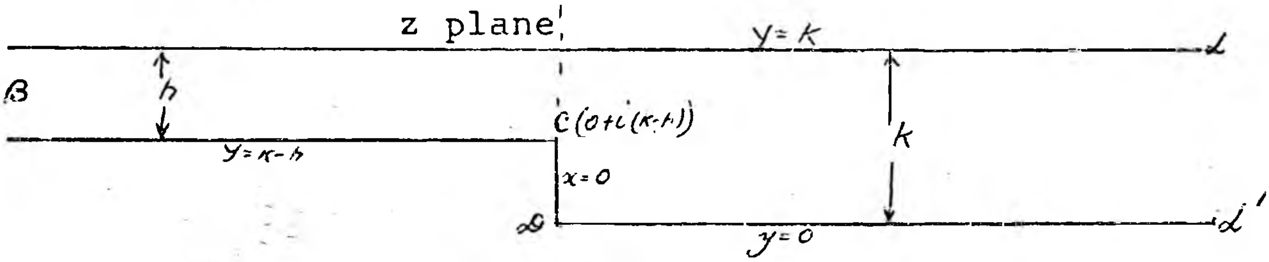
SECTION 7.1

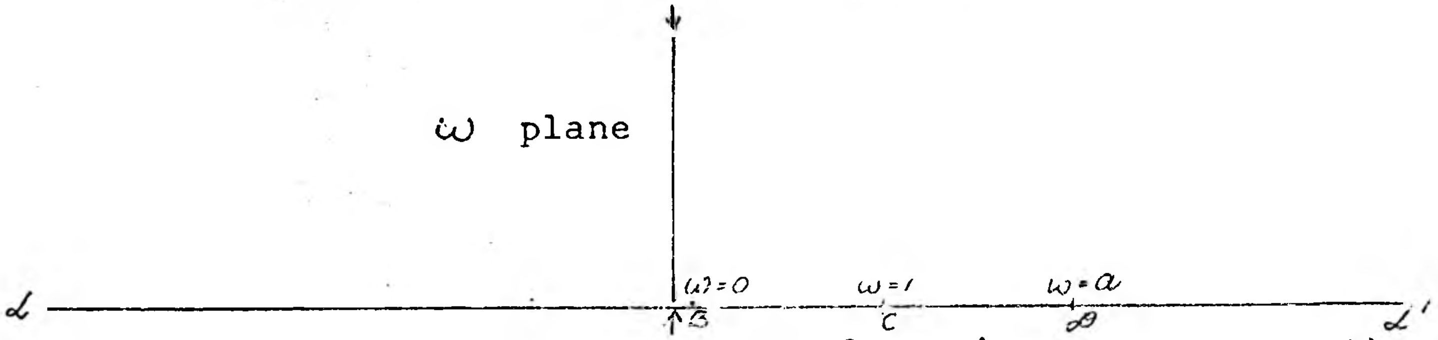
TRANSFORMATION FUNCTION AND DERIVATION OF SIMULTANEOUS EQUATIONS.

SECTION 7.1.1

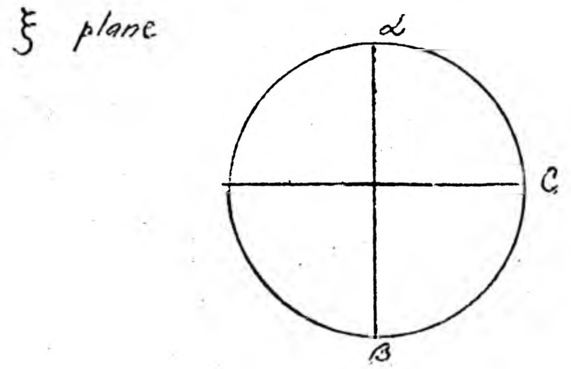
TRANSFORMATION FUNCTION

Consider the case of an infinitely long 'beam' under plain strain with width h on one side of a discontinuity and k on the other. Then map this on to the half-plane in the ω plane and this on to the unit circle in the ξ plane.





See Dictionary of Conformal Transformation (ref. 4 p.161)



From Schwarz-Christoffel

$$\frac{dz}{d\omega} = \frac{k(\omega-1)^{\frac{1}{2}}}{\pi\omega(\omega-a)^{\frac{1}{2}}} \qquad \text{where } a = \left(\frac{k}{h}\right)^2 > 1 \;;\; k > 0$$

$$z = \frac{k}{\pi}\cosh^{-1}\left(\frac{2\omega-a-1}{a-1}\right) - \frac{k}{\pi\sqrt{a}}\cosh^{-1}\left(\frac{(a+1)\omega-2a}{(a-1)\omega}\right)$$

and

$$\omega = \frac{i+\xi}{1+i\xi}$$

Let $\frac{k}{\pi} = A$

Then

$$\frac{dz}{d\xi} = \frac{d\omega}{d\xi}\cdot\frac{dz}{d\omega}$$

$$= \frac{2}{(1+i\xi)^2}\cdot\frac{A}{\omega}\cdot\frac{(\omega-1)^{1/2}}{(\omega-a)^{1/2}}$$

$$= \frac{2A}{(1+i\xi)^2}\cdot\frac{\left(1-\frac{i+\xi}{1+i\xi}\right)^{1/2}}{\frac{i+\xi}{1+i\xi}\cdot\left(a-\frac{i+\xi}{1+i\xi}\right)^{1/2}}$$

$$= \frac{2A}{(1+i\xi)(i+\xi)}\cdot\frac{(1+i\xi-i-\xi)^{1/2}}{(a+ai\xi-i-\xi)^{1/2}}$$

$$= \frac{2A.(1-i)^{1/2}(1-\xi)^{1/2}}{i(1+\xi^2).(a-i)^{1/2}\left(1-\frac{1-ai}{a-i}\xi\right)^{1/2}}$$

$$= -\frac{2Ai(1-i)^{1/2}(1-\xi)^{1/2}}{(a-i)^{1/2}(1+\xi^2)\left(1-\frac{1-ai}{a-i}\xi\right)^{1/2}}$$

$$\frac{dz}{d\xi} = -B\frac{(1-\xi)^{1/2}}{(1+\alpha\xi)^{1/2}(1+\xi^2)} \qquad \ldots \quad (7.1)$$

where $B = \frac{2Ai(1-i)^{1/2}}{(a-i)^{1/2}}$

and $\alpha = -\frac{1-ai}{(a-i)} = -\frac{a-a^2i+i+a}{(1+a^2)} = \frac{-2a+i(a^2-1)}{1+a^2}$

(Note that the magnitude of α = 1)

As a first case we assume $a = 4$ and $A = 2$

(this latter is completely arbitrary) i.e. $h=\pi, k=2\pi$

Then $B = .62533 + 2.25764i$

and $\alpha = -.47059 + .88235i$

Section 7.1.2 Derivation of u_n and hence set of simultaneous equations.

Expanding (7.1)

$$\frac{dz}{d\xi} = -B\left(1-\frac{1}{2}\xi-\frac{1}{2^2 2!}\xi^2-\frac{1.3}{2^3 3!}\xi^3-\ldots-\frac{1.3.5\ldots(2r-3)}{2^r r!}\xi^r-\ldots\right)$$

$$.\left(1-\frac{1}{2}\alpha\xi+\frac{1.3}{2^2 2!}(\alpha\xi)^2-\frac{1.3.5}{2^3 3!}(\alpha\xi)^3+\ldots+(-1)^r\frac{1.3.5\ldots(2r-1)}{2^r r!}\alpha^r\xi^r+\ldots\right)$$

$$.\left(1-\xi^2+\xi^4-\xi^6+\ldots+(-1)^r\xi^{2r}+\ldots\right)$$

$$\frac{dz}{d\xi} = -B\Big(1+\xi\left(-\frac{1}{2}-\frac{1}{2}\alpha\right)+\xi^2\left(-\frac{1}{2^2 2!}+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\alpha\right)+1.\left(\frac{1.3}{2^2 2!}\alpha^2\right)\right)$$

$$+\xi^3\left(-\frac{1.3}{2^3 3!}+\left(-\frac{1}{2^2 2!}\right)\left(-\frac{1}{2}\alpha\right)-\frac{1}{2}\left(\frac{1.3}{2^2 2!}\alpha^2\right)+1.\left(-\frac{1.3.5}{2^3 3!}\alpha^3\right)\right)$$

$$+\xi^4\left(-\frac{1.3.5}{2^4 4!}1-\frac{1.3}{2^3 3!}\left(-\frac{1}{2}\alpha\right)+\left(-\frac{1}{2^2 2!}\right)\left(\frac{1.3}{2^2 2!}\alpha^2\right)-\frac{1}{2}\left(-\frac{1.3.5}{2^3 3!}\alpha^3\right)+1\frac{1.3.5.7}{2^4 4!}\alpha^4\right)+\ldots$$

$$+\xi^r\Big\{-\frac{1.3.5\ldots(2r-3)}{2^r.r!}.1-\frac{1.3.5\ldots(2r-5)}{2^{r-1}(r-1)!}\left(-\frac{1}{2}\alpha\right)-\frac{1.3.5\ldots(2r-7)}{2^{r-2}(r-2)!}.$$

$$\left(\frac{3}{2^2 2}\alpha^2\right)-\ldots$$

$$-\frac{1.3.5\ldots(2r-3-2s)}{2^{r-s}(r-s)!}(-1)^s\frac{1.3.5\ldots(2s-1)}{2^s s!}\alpha^s+\ldots$$

$$+\frac{1}{2}(-1)^{r-1}\frac{1.3.5\ldots(2r-3)}{2^{r-1}(r-1)!}\alpha^{r-1}+(-1)^r\frac{1.3.5\ldots(2r-1)}{2^r r!}\alpha^r\Big\}+\ldots\Big).$$

$$\left(1-\xi^2+\xi^4-\xi^6+\ldots+(-1)^r\xi^{2r}+\ldots\right) \qquad \ldots \text{ eqn. } (7.2)$$

$$\frac{dz}{d\xi} = -B\Big[1+\xi\left(-\tfrac{1}{2}-\tfrac{1}{2}\alpha\right)+\xi^{2}\left(\left\{-\tfrac{1}{2^{2}2!}\cdot 1-\tfrac{1}{2}\left(-\tfrac{1}{2}\alpha\right)+\tfrac{1.3}{2^{2}2!}\alpha^{2}\right\}-1\right)+\xi^{3}\left(\left\{-\tfrac{1.3}{2^{3}3!}+\left(-\tfrac{1}{2^{2}2!}\right)\left(-\tfrac{1}{2}\alpha\right)\right.\right.$$

$$-\tfrac{1}{2}\left(\tfrac{1.3}{2^{2}2!}\alpha^{2}\right)+1\left(-\tfrac{1.3.5}{2^{3}3!}\alpha^{3}\right)\Big\}\{-1\}^{0/2}+\left\{-\tfrac{1}{2}-\tfrac{1}{2}\alpha\right\}\{-1\}^{2/2}\Big)+\xi^{4}\Big(\Big\{-\tfrac{1.3.5}{2^{4}4!}\cdot 1+\tfrac{1.3}{2^{3}3!}\left(-\tfrac{1}{2}\alpha\right)$$

$$-\tfrac{1}{2^{2}2!}\left(\tfrac{1.3}{2^{2}2!}\alpha^{2}\right)-\tfrac{1}{2}\left(-\tfrac{1.3.5}{2^{3}3!}\alpha^{3}\right)+1\cdot\tfrac{1.3.5.7}{2^{4}4!}\alpha^{4}\Big\}\{-1\}^{0/2}+\left\{-\tfrac{1}{2^{2}2!}\cdot 1-\tfrac{1}{2}\left(-\tfrac{1}{2}\alpha\right)+1\left(\tfrac{1.3}{2^{2}2!}\alpha^{2}\right)\right\}$$

$$\{-1\}^{2/2}+\{-1\}^{4/2}\Big)+\xi^{5}\Big(\Big\{-\tfrac{1.3.5.7}{2^{5}5!}\cdot 1-\tfrac{1.3.5}{2^{4}4!}\left(-\tfrac{1}{2}\alpha\right)-\tfrac{1.3}{2^{3}3!}\left(\tfrac{1.3}{2^{2}2!}\alpha^{2}\right)-\tfrac{1}{2^{2}2!}\left(-\tfrac{1.3.5}{2^{3}3!}\alpha^{3}\right)$$

$$-\tfrac{1}{2}\left(\tfrac{1.3.5.7}{2^{4}4!}\alpha^{4}\right)+1\left(-\tfrac{1.3.5.7.9}{2^{5}5!}\alpha^{5}\right)\Big\}\{-1\}^{0/2}+\Big\{-\tfrac{1.3}{2^{3}3!}+\left(-\tfrac{1}{2^{2}2!}\right)\left(-\tfrac{1}{2}\alpha\right)-\tfrac{1}{2}\left(\tfrac{1.3}{2^{2}2!}\alpha^{2}\right)$$

$$+1\left(-\tfrac{1.3.5}{2^{3}3!}\alpha^{3}\right)\Big\}\{-1\}^{2/2}+\left\{\tfrac{1}{2}-\tfrac{1}{2}\alpha\right\}\{-1\}^{4/2}\Big)\;+\;\cdots$$

$$+\xi^{r}\Big(\Big\{-\frac{1.3.5\ldots(2r-3)}{2^{r}r!}-\frac{1.3.5\ldots(2r-5)}{2^{r-1}(r-1)!}\left(-\tfrac{1}{2}\alpha\right)-\frac{1.3.5\ldots(2r-7)}{2^{r-2}(r-2)!}\left(\frac{3}{2^{2}2!}\alpha^{2}\right)-\frac{1.3.5\ldots(2r-3-2s)}{2^{r-s}(r-s)!}(-1)^{s}\;\cdot$$

$$\cdot\frac{1.3.5\ldots(2s-1)}{2^{s}s!}\alpha^{s}-\cdots-\tfrac{1}{2}(-1)^{r-1}\frac{1.3.5\ldots(2r-3)}{2^{r-1}(r-1)!}\alpha^{r-1}+(-1)^{r}\frac{1.3.5\ldots(2r-1)}{2^{r}r!}\alpha^{r}\Big\}\{-1\}^{0/2}$$

$$+\Big\{-\frac{1.3.5\ldots(2r-7)}{2^{r-2}(r-2)!}-\frac{1.3.5\ldots(2r-9)}{2^{r-3}(r-3)!}\left(-\tfrac{1}{2}\alpha\right)-\frac{1.3.5\ldots(2r-11)}{2^{r-4}(r-4)!}\left(\frac{3}{2^{2}2!}\alpha^{2}\right)-\cdots-\frac{1.3.5\ldots(2r-7-2s)}{2^{r-2-s}(r-2-s)!}\;\cdot$$

$$(-1)^{s}\frac{1.3.5\ldots(2s-1)}{2^{s}s!}\alpha^{s}-\cdots-\tfrac{1}{2}(-1)^{r-3}\frac{1.3.5\ldots(2r-7)}{2^{r-3}(r-3)!}\alpha^{r-3}+1(-1)^{r-2}\frac{1.3.5\ldots(2r-5)}{2^{r-2}(r-2)!}\alpha^{r-2}\Big\}\{-1\}^{2/2}$$

$$+\cdots+\Big\{-\frac{1.3.5\ldots(2r-t-3)}{2^{r-t}(r-t)!}-\frac{1.3.5\ldots(2r-t-5)}{2^{r-t-1}(r-t-1)!}\left(-\tfrac{1}{2}\alpha\right)-\frac{1.3.5\ldots(2r-t-7)}{2^{r-t-2}(r-t-2)!}\left(\frac{3}{2^{2}2!}\alpha^{2}\right)-\cdots$$

$$-\frac{1.3.5\ldots(2r-t-3-2s)}{2^{r-t-s}(r-t-s)!}(-1)^{s}\frac{1.3.5}{2^{s}s!}(2s-1)\alpha^{s}-\cdots-\tfrac{1}{2}(-1)^{r-1-t}\frac{1.3.5\ldots(2r-t-3)}{2^{r-t-1}(r-t-1)!}\alpha^{r-t-1}$$

$$+1(-1)^{r-t}\frac{1.3.5\ldots(2r-t-1)}{2^{r-t}(r-t)!}\alpha^{r-t}\Big\}\{-1\}^{\frac{t}{2}}+\cdots\Big)\;+\;\cdots\Big]\quad\ldots\text{eqn}(7.3)$$

$$= \sum_{r=0}^{\infty}\sum_{\substack{t=0\\ t=0,2,4,\ldots}}^{r}\sum_{s=0}^{r-t}\left(- \frac{1.3.5\ldots(2\overline{r-t}-3-2s)}{2^{r-t-s}\,(r-t-s)!}(-1)^s \frac{1.3.5\ldots(2s-1)\alpha^s}{2^s\, s!}\,(-1)^{t/2}\right.$$

provided when $(2\overline{r-t}-3-2s)$ or $(2s-1)$ negative, the product $1.3.5\ldots = 1$

For this configuration i.e. $\alpha = -.47059 + .88235i$

$$\frac{dz}{d\xi} = -Bi\left(-.441\xi - .091\xi^2 + .684\xi^3 + .395\xi^4 - .588\xi^5 - .544\xi^6 + .386\xi^7 + \ldots\right)$$
$$-B\left(1 - .265\xi - 1.452\xi^2 - .034\xi^3 + 1.476\xi^4 + .258\xi^5 - 1.320\xi^6 - .325\xi^7 \ldots\right)$$

Refer programme CALC as described on p.74 for details of the values of the coefficients of ξ^n.

Using value of B

$$\frac{dz}{d\xi} = \pm(.625 + 2.258i)\left(1 + \xi(-.265 - .441i) + \xi^2(-1.452 - .091i) + \xi^3(-.034 + .684i) + \xi^4(1.476 + .395i) + \xi^5(.258 - .588i) \ldots\right)$$

Integrating, and taking $B = +(.625 + 2.258i)$,

$$Z = U_0 + (-.625 - 2.258i)\xi + \xi^2(-.415 + .437i) + \xi^3(.234 + 1.111i) + \xi^4(.391 - .088i) + \xi^5(-.006 - .716i) \ldots \qquad \text{Eqn. } (7.4)$$

and $U_0 = Z$ when $\xi = 0$ i.e. when $\omega = i$

and $Z(\omega = i) = \frac{k}{\pi}\cosh^{-1}\left(\frac{2i-a-1}{a-1}\right) - \frac{k}{\pi\sqrt{a}}\cosh^{-1}\frac{i(a+1)-2a}{(a-1)i}$

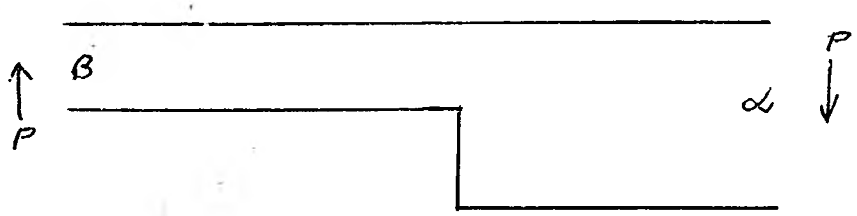
$= U_0 = .595 + 4.365i$ when $a=4$, $A=2$

Now if $Z = U_0 + U_1\xi + U_2\xi^2 + U_3\xi^3 + \ldots$

from eqn. (7.4) we know $U_1, U_2, U_3, \ldots$ and can substitute these values in eqns. 3.3 to 3.6 on p. 14 and calculate $b_0, b_1, b_2, \ldots$

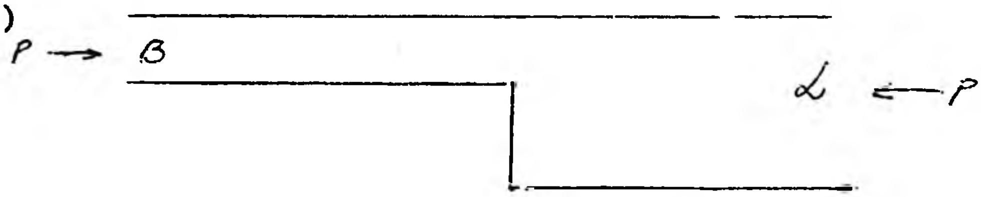
Consider two loading cases :

LOADING (1)



The right hand sides are then those calculated for the two previous shapes.

LOADING (2)



The right hand sides are then -i times those for case 1.

Letting $u_n = \theta_n + i\phi_n$

and $b_n = \alpha_n + i\beta_n$, eqns. 3.3 to 3.6 (p. 14) become :

$$
\begin{aligned}
&(\theta_1+i\phi_1)(\alpha_0+i\beta_0) + \Big\{(\theta_1+i\phi_1)(\alpha_0-i\beta_0) + (\theta_2+i\phi_2)(\alpha_1-i\beta_1) \\
&\quad + (\theta_3+i\phi_3)(\alpha_2-i\beta_2) + \dots \Big\} = \begin{cases} -4/\pi & \dots(1) \\ +4i/\pi & \dots(2) \end{cases} && (7.10) \\
&(\theta_1+i\phi_1)(\alpha_1+i\beta_1) + 2(\theta_2+i\phi_2)(\alpha_0+i\beta_0) + 2\Big\{(\theta_2+i\phi_2)(\alpha_0-i\beta_0) \\
&\quad + (\theta_3+i\phi_3)(\alpha_1-i\beta_1) + \dots \Big\} = 0 && (7.11) \\
&(\theta_1+i\phi_1)(\alpha_2+i\beta_2) + 2(\theta_2+i\phi_2)(\alpha_1+i\beta_1) + 3(\theta_3+i\phi_3)(\alpha_0+i\beta_0) \\
&\quad + 3\Big\{(\theta_3+i\phi_3)(\alpha_0-i\beta_0) + (\theta_4+i\phi_4)(\alpha_1-i\beta_1) + \\
&\quad \dots \Big\} = \begin{cases} 4/\pi & \dots(1) \\ -4i/\pi & \dots(2) \end{cases} && (7.12) \\
&(\theta_1+i\phi_1)(\alpha_3+i\beta_3) + 2(\theta_2+i\phi_2)(\alpha_2+i\beta_2) + 3(\theta_3+i\phi_3)(\alpha_1+i\beta_1) \\
&\quad + 4(\theta_4+i\phi_4)(\alpha_0+i\beta_0) + 4\Big\{(\theta_4+i\phi_4)(\alpha_0-i\beta_0) \\
&\quad + (\theta_5+i\phi_5)(\alpha_1-i\beta_1) + \dots \Big\} = 0 \quad \text{etc.} && (7.13)
\end{aligned}
$$

SECTION 7.2

RE-ORGANIZATION OF EQUATIONS .

Let

$$\alpha_0 = x_1 \qquad \beta_0 = x_2$$
$$\alpha_1 = x_3 \qquad \beta_1 = x_4$$
$$\alpha_2 = x_5 \qquad \beta_2 = x_6$$

and

$$\theta_1 = A_1 \qquad \phi_1 = A_2$$
$$\theta_2 = A_3 \qquad \phi_2 = A_4$$
$$\theta_3 = A_5 \qquad \phi_3 = A_6$$

Then the equations become, separating real and imaginary parts

$$(7.10.1)\quad \left\{\begin{array}{l} A_1x_1 - A_2x_2 \\ \qquad + \left(A_1x_1+A_2x_2+A_3x_3+A_4x_4+\ldots\right) = \begin{cases} -4/\pi & \ldots(1) \\ 0 & \ldots(2) \end{cases} \\ A_2x_1 + A_1x_2 \\ \qquad + \left(A_2x_1-A_1x_2+A_4x_3-A_3x_4+\ldots\right) = \begin{cases} 0 & \ldots(1) \\ 4/\pi & \ldots(2) \end{cases} \end{array}\right.$$

$$(7.11.1)\quad \left\{\begin{array}{l} A_1x_3 - A_2x_4 + 2(A_3x_1-A_4x_2) \\ \qquad +2\left(A_3x_1+A_4x_2+A_5x_3+A_6x_4+\ldots\right) = 0 \\ A_2x_3 + A_1x_4 + 2(A_4x_1+A_3x_2) \\ \qquad +2\left(A_4x_1-A_3x_2+A_6x_3-A_5x_4+\ldots\right) = 0 \end{array}\right.$$

and so on.

And, by calculation and also from FORTRAN programme CALC,

$A_1 = -.625 \qquad A_2 = -2.258 \qquad A_3 = -.415 \qquad A_4 = .437$

$A_5 = .234 \qquad A_6 = 1.111 \qquad A_7 = .391 \qquad A_8 = -.088$ etc.

Refer to the printer output of CALC, which attempts to solve the above infinite set of simultaneous equations by the simple iteration method, for the relevant values.

CALC (case 1.1)

Lines 1 to 4 represent the real parts of $\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{49}$

... 5 to 8 ... . imag. .. . . . .

Lines 17 to 20 represent real parts of λ_1, λ_2-1, $\lambda_3-\lambda_1$, $\lambda_4-\lambda_2+1$, $\lambda_5-\lambda_3+\lambda_1$, $\lambda_6-\lambda_4+\lambda_2-1, \ldots$, $\lambda_{49}-\lambda_{47}+\lambda_{45}-\ldots+\lambda_1$

Lines 21 to 24 represent the corresponding imaginary parts.

where $\lambda_1=-\frac{1}{2}-\frac{1}{2}\alpha \qquad = \text{coeff. of } \xi^1 \text{ in eqn.}(7.2)$

$$\lambda_2=-\frac{1}{2^2 2!}+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\alpha\right)+1\left(\frac{1\cdot 3}{2^2 2!}\alpha^2\right) = \text{coeff. of } \xi^2$$

$$\lambda_3=-\frac{1\cdot 3}{2^3 3!}+\left(-\frac{1}{2^2 2!}\right)\left(-\frac{1}{2}\alpha\right)-\frac{1}{2}\left(\frac{1\cdot 3}{2^2 2!}\alpha^2\right)+1\left(-\frac{1\cdot 3\cdot 5}{2^3 3!}\alpha^3\right)$$

$$= \text{coeff. of } \xi^3 \quad \text{etc.}$$

Line 25 represents the real and imaginary parts of -B.

Lines 26 to 33 represent the values $A_1, A_2, A_3, \ldots, A_{98}$.

It is then necessary to calculate the coeffs. of x_1, x_2, x_3, ... in both equations (7.10.1), both equations (7.11.1), etc.. These are listed from coeffs. of x_1 to x_{30} for 30 eqns. which takes us to the line on the third page of the output marked END OF UNADJUSTED COEFFS.. If these coeffs. are called

$$\begin{array}{llll} B_{1\,1} & B_{2\,1} & \ldots & B_{30\,1} \\ B_{1\,2} & B_{2\,2} & \ldots & B_{30\,2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{1\,30} & B_{2\,30} & \ldots & B_{30\,30} \end{array},$$

then for our set of eqns. to 'converge', $B_{1\,1}$, $B_{2\,2}$, $B_{3\,3}$, $\ldots B_{30\,30}$ must predominate in their respective rows.

Obviously, this is not the case.

However, if we add :

the second eqn. to the sixth and call it eqn. (8.5)

the first to the fifth . . . . (8.6)

the fourth to the eighth . . . . (8.7)

the third to the seventh . . . . (8.8)

. . . . . .

the twenty-sixth to the thirtieth . . . . (8.29)

the twenty-fifth to the twenty-ninth . . . (8.30) ,

equations (8.5) to (8.30) are then suitable and the right hand sides are 0. It is then necessary to obtain suitable equations (8.1), (8.3), and (8.4) (eqn.(8.2) is not required since x_2 is indeterminate) . The best method here is not obvious, nor is it very good.

		r.h.s. LOADING (1)	r.h.s. (2)
Case 1.			
eqn.8.1 =	first eqn.	$-4/\pi$	0
eqn.8.3 =	first eqn. + fourth eqn.	$-4/\pi$	0
eqn.8.4 =	third eqn. + fourth eqn.	0	0
Case 2.			
eqn.8.1 =	sixth eqn. - first eqn.	$+4/\pi$	$-4/\pi$
eqn.8.3 =	first eqn. - eighth eqn.	$-4/\pi$	0
eqn.8.4 =	third eqn. + fourth eqn.	0	0

(The programme uses $\pi/4$ times the real r.h.s.'s i.e. 0 or ±1)

Case 1 is obviously useless for loading (2), since $b_n = 0$ for all n is a solution .

Comparing the coefficients of equations (8.1), (8.3), and (8.4) for each case shows that, in Case 1, eqn. (8.3) is rather unsatisfactory due to the relatively low absolute value of x_3, whilst in Case 2, it is much better, though still far from ideal.

SECTION 7.3

RESULTS OF ITERATION PROGRAMME (CALC)

For LOADING (1), Case 1 re-organization was tried first but the convergence was not very satisfactory. Case (2) was tried next and gave much better convergence but produced entirely different results. Initially, both cases were tried using only 30 iteration loops and thus calculating x_1 to x_{30}, case 2 was later run with 50 loops calculating x_1 to x_{50}.

For case 1 only the computer output of the re-organized eqns.(8.1),(8.3),(8.4) and the printout of the iteration loops is shown. For case 2 the following cases were run on the computer (only parts of the output are shown, since the whole output would involve much repetition and an excessively large number of pages) :-

LOADING (1)

1.1 30 loops with $a = 4.00$ i.e. $k/h = 2$

(the whole of the programme is shown)

1.2 50 loops $a = 4.00$

(only the printout of the iteration loops 31 to 50 shown)

1.3 30 loops $a = 1.21$ i.e. $k/h = 1.1$

(the whole of the output of the programme is shown)

LOADING (2)

2.1 50 loops $a = 4.00$

(iteration loops only)

2.2 50 loops $a = 1.21$ (loops only)

The case when a = 16.00 i.e. k/h = 4.0 was also tried but did not appear to produce a solution for either loading case.

Case 1 re-organization results, as well as being useless for LOADING case (2) and giving less satisfactory results for LOADING case (1), are shown later to be incompatible with the results of the successive approximation method and can therefore be ignored.

In other cases the convergence is quite satisfactory, especially for loading case (2).

There seems to be some correspondence between groups of 8 iterations (this is further borne out later, and is to be expected since the r.h.s.'s of the original equations repeats every 8 equations) and so the average of the last 8 values calculated for each x_n is found. When there are not 8 values(x_{43} to x_{50}), the average of those iterations available is found.

Case 2.1

In this case the convergence is very good since x_n appears to approach 0 as n approaches infinity. This means that the simple iteration method has all the advantages of the successive approximation method, which is therefore of no further use. The average of the last 8 iterations is shown in each case.

x_1 = -.2174		x_3 = .0008	x_4 = -.0633
x_5 = -.0985	x_6 = -.0913	x_7 = -.1133	x_8 = -.0323
x_9 = -.0828	x_{10} = -.0076	x_{11} = -.0631	x_{12} = -.0207
x_{13} = -.0701	x_{14} = -.0268	x_{15} = -.0747	x_{16} = -.0140
x_{17} = -.0638	x_{18} = -.0060	x_{19} = -.0548	x_{20} = -.0123
x_{21} = -.0581	x_{22} = -.0168	x_{23} = -.0618	x_{24} = -.0111
x_{25} = -.0568	x_{26} = -.0057	x_{27} = -.0510	x_{28} = -.0079
x_{29} = -.0515	x_{30} = -.0106	x_{31} = -.0529	x_{32} = -.0085
x_{33} = -.0503	x_{34} = -.0061	x_{35} = -.0427	x_{36} = -.0075
x_{37} = -.0484	x_{38} = -.0083	x_{39} = -.0487	x_{40} = -.0059
x_{41} = -.0458	x_{42} = -.0046	x_{43} = -.0442	x_{44} = -.0068
x_{45} = -.0459	x_{46} = -.0087	x_{47} = -.0469	x_{48} = -.0065
x_{49} = -.0427	x_{50} = -.0057		

Case 1.2

The convergence is not as good as for loading case (2) which is to be expected since x_n, for odd n, appears to approach about .50 as n approaches infinity. That is

$b_n \to .50 + 0i$ as $n \to \infty$.

Again the table shows the averages of the last 8 iterations.

x_1 = .5182		x_3 =2.8676	x_4 = .5441
x_5 = .9231	x_6 = .9851	x_7 = .6270	x_8 = .4501
x_9 = .9638	x_{10} = .0943	x_{11} =1.0386	x_{12} = .1878
x_{13} = .7677	x_{14} = .2833	x_{15} = .6307	x_{16} = .1838
x_{17} = .7428	x_{18} = .0957	x_{19} = .7973	x_{20} = .1337
x_{21} = .6766	x_{22} = .1645	x_{23} = .5901	x_{24} = .1126
x_{25} = .6383	x_{26} = .0734	x_{27} = .6733	x_{28} = .1007
x_{29} = .6047	x_{30} = .1207	x_{31} = .5512	x_{32} = .0880
x_{33} = .5812	x_{34} = .0631	x_{35} = .6016	x_{36} = .0803
x_{37} = .5549	x_{38} = .0936	x_{39} = .5162	x_{40} = .0731
x_{41} = .5343	x_{42} = .0562	x_{43} = .5575	x_{44} = .0737
x_{45} = .5340	x_{46} = .0897	x_{47} = .5074	x_{48} = .0698
x_{49} = .5005	x_{50} = .0666		

Cases 1.3 and 2.2

Convergence is similar to that of the $a = 4.00$ cases. See the programme printouts for the values.

It would be expected that a successive approximation method, where the initial values of b_n for $n > 50$ is about .50, would be useful for the loading case (1) and this method has produced results which agree fairly closely with the PURE ITERATION method.

I believe the results produced by this method to be superior to those of PURE ITERATION, but the calculations are long and tedious and the improvement only marginal, so these calculations are done in the appendix (APP. D)

The results of programme COSI(2) Case 2A are expected to be the best and are shown in the table APP. D p. 131 under 'new x'.

It would seem worthwhile to make more loops in programme CALC and calculate more x's, say up to x_{100}. A serious restriction is that the core requirements of the coefficients B increase as the square of the number of x's calculated. As it stands CALC has two arrays of this size, B and BA, but this could be reduced to one. This procedure may produce an improvement as worthwhile as the SUCCESSIVE APPROXIMATION method for loading Case 1.

SECTION 7.4

BRIEF DISCUSSION OF METHOD OF CALCULATING na_n .

To calculate the a_n, knowing b_n, we again use the eqns. :

$$na_n + n(\bar{u}_0 b_n + \bar{u}_1 b_{n+1} + \dots \qquad) = Q_n \qquad \text{all } n$$

For loading case (2), the series may be summable as it stands, but for case (1), some adjustment is necessary :

Let $b_n = C + y_n$ where C is a constant which b_n appears to approach for high n. Then the y_n approach 0 as $n \to \infty$.

Then $$na_n + n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots) + nC(\bar{u}_0 + \bar{u}_1 + \bar{u}_2 + \dots \quad) = Q_n$$

But when $\xi = 1$, $z = u_0 + u_1 + u_2 + \dots \qquad = i(k-h)$

so $\bar{u}_0 + \bar{u}_1 + \bar{u}_2 + \dots \qquad = -i(k-h) = -i\pi$ when $a = 4.00$

then $$na_n = Q_n - n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots) + nCi(k-h)$$

For high n, $(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots)$ should be very small and $n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots)$ may or may not approach 0.

However, $\qquad na_n \simeq nCi(k-h)$

Then $$\omega_z'' = (a_1 + 2a_2\xi + 3a_3\xi^2 + \dots)\left(-\frac{(1+\alpha\xi)^{1/2}(1+\xi^2)}{B(1-\xi)^{1/2}}\right)$$

and $a_1 + 2a_2\xi + 3a_3\xi^2 + \dots.$

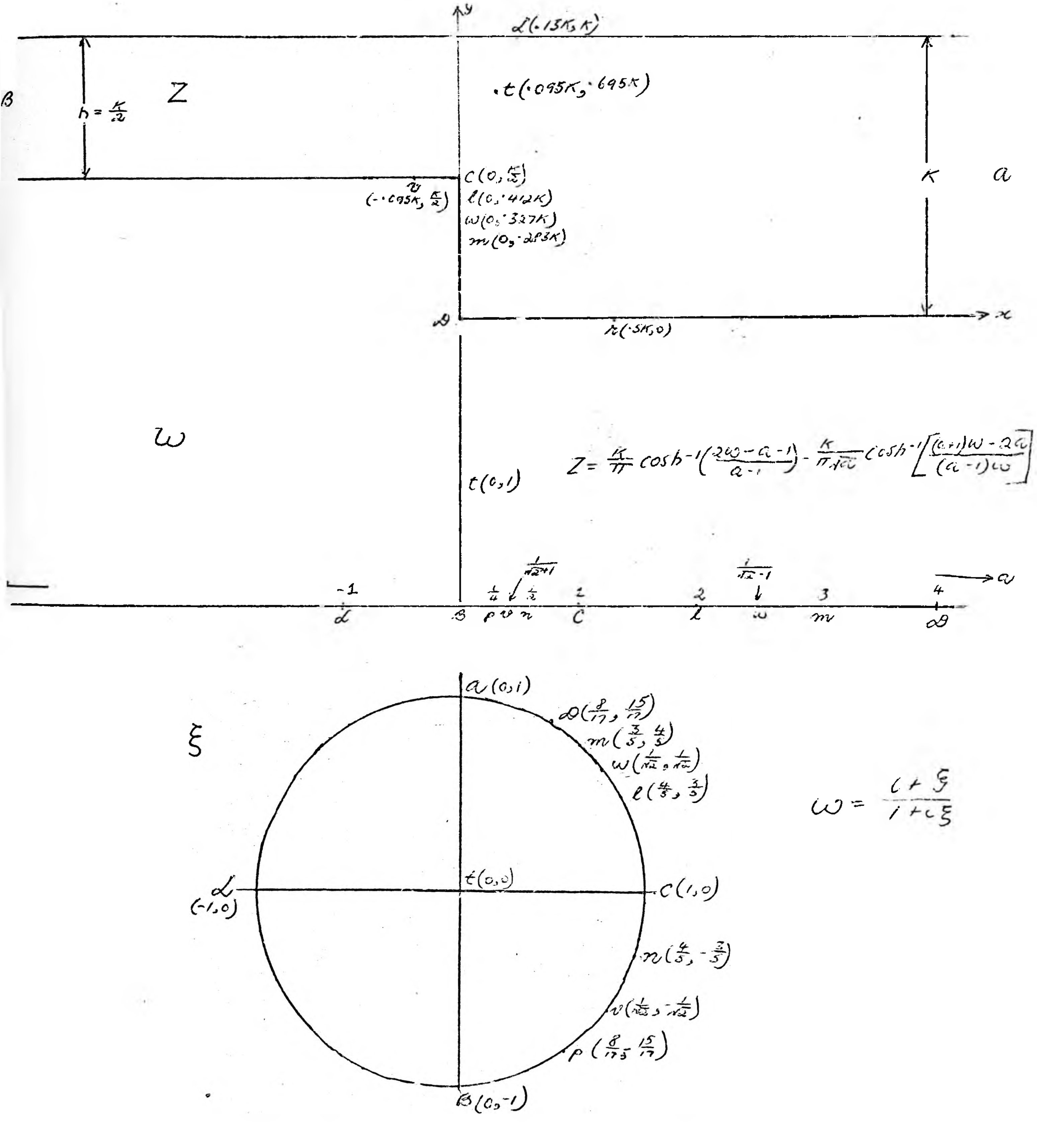
$$= Ci(k-h)(1+2\xi+3\xi^2+\dots) + \sum_{n=1}^{\infty}\{Q_n - n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots)\}\xi^{n-1}$$

$$= \frac{Ci(k-h)}{(1-\xi)^2} + \sum_{n=1}^{\infty}\{Q_n - n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots)\}\xi^{n-1}$$

within the boundary, and this can be calculated away from the boundary provided $n(\bar{u}_0 y_n + \bar{u}_1 y_{n+1} + \dots)$ approaches a finite constant as n approaches infinity. Considering the relatively few terms which we can evaluate in the infinite series, this would have to approach a small constant quickly if we were to obtain accurate results. These calculations have not been made.

SECTION 7.5

SOME VALUES OF $\hat{x}x + \hat{y}y$.



For loading CASE 2 and a = 4.00 i.e. $k = 2\pi$, $h = \pi$

$$
\begin{aligned}
\Omega'/(4/\pi) = & -.2174 + .0008\xi - .0985\xi^2 - .1133\xi^3 - .0828\xi^4 \\
& - .0631\xi^5 - .0701\xi^6 - .0747\xi^7 - .0638\xi^8 - .0548\xi^9 \\
-.0581\xi^{10} & - .0610\xi^{11} - .0568\xi^{12} - .0510\xi^{13} - .0515\xi^{14} \\
& - .0529\xi^{15} - .0503\xi^{16} - .0477\xi^{17} - .0484\xi^{18} \\
& - .0487\xi^{19} - .0458\xi^{20} - .0442\xi^{21} - .0459\xi^{22} \\
& - .0469\xi^{23} - .0427\xi^{24} \\
- i \{ & .0633\xi + .0913\xi^2 + .0323\xi^3 + .0076\xi^4 + .0207\xi^5 \\
& + .0268\xi^6 + .0140\xi^7 + .0060\xi^8 + .0123\xi^9 \\
& + .0168\xi^{10} + .0111\xi^{11} + .0057\xi^{12} + .0079\xi^{13} \\
& + .0106\xi^{14} + .0085\xi^{15} + .0061\xi^{16} + .0075\xi^{17} \\
& + .0083\xi^{18} + .0059\xi^{19} + .0046\xi^{20} + .0068\xi^{21} \\
& + .0087\xi^{22} + .0065\xi^{23} + .0057\xi^{24}
\end{aligned}
$$

<u>At $\xi = -i$</u> (β on diag.)

$$R(\Omega'/\tfrac{4}{\pi}) \simeq -.210 = (\widehat{xx} + \widehat{yy})\frac{\pi}{4}$$

<u>At $\xi = +i$</u> (a on diag.)

$$R(\Omega'/\tfrac{4}{\pi}) \simeq -.123$$

<u>$\xi = 0$</u> (t on diag.)

$$R(\Omega'/\tfrac{4}{\pi}) = -.2174$$

<u>$\xi = -1$</u> (α on diag.)

$$R(\Omega'/\tfrac{4}{\pi}) \simeq \sim .229$$

$\xi = 1$ (C on diag.)

In this case, all terms after the second are negative and the rate of decrease of the terms appears to be less than that of the harmonic series. Therefore, the value of $\widehat{xx}+\widehat{yy}$

at this point appears to be $-\infty$.

For loading Case 2 and a = 1.21 i.e. $k = 2\pi, h = 2\pi/1.1$

$\frac{\pi}{4}(\widehat{xx} + \widehat{yy})$ at $\xi = -i \quad \simeq -.135$

$\frac{\pi}{4}(\widehat{xx} + \widehat{yy})$ at $\xi = +i \quad \simeq -.125$

For loading Case 1 and a = 4.00 using results of programme COSI(2)-Case 2A

$$\Omega'/(\tfrac{4}{\pi}) = .500\,(1 + \xi + \xi^2 + \xi^3 + \ldots)$$
$$+ .007 + 2.294\xi + .411\xi^2 + .125\xi^3 + .455\xi^4 + .517\xi^5$$
$$+ .260\xi^6 + .131\xi^7 + .239\xi^8 + .291\xi^9 + .174\xi^{10} + .092\xi^{11}$$
$$+ .142\xi^{12} + .179\xi^{13} + .115\xi^{14} + .061\xi^{15} + .088\xi^{16}$$
$$+ .112\xi^{17} + .069\xi^{18} + .033\xi^{19} + .051\xi^{20} + .066\xi^{21}$$
$$+ .035\xi^{22} + .009\xi^{23} + .025\xi^{24} + \ldots$$
$$+ i\,(.521\xi + .951\xi^2 + .431\xi^3 + .089\xi^4 + .183\xi^5$$
$$+ .275\xi^6 + .177\xi^7 + .089\xi^8 + .125\xi^9 + .155\xi^{10}$$
$$+ .107\xi^{11} + .071\xi^{12} + .097\xi^{13} + .113\xi^{14} + .079\xi^{15}$$
$$+ .056\xi^{16} + .076\xi^{17} + .089\xi^{18} + .066\xi^{19} + .049\xi^{20}$$
$$+ .062\xi^{21}\ .072\xi^{22} + .056\xi^{23} + .043\xi^{24} + \ldots)$$

$\underline{\xi = -i}$

$$R(\Omega'/\tfrac{4}{\pi}) = .250 - .069 + .179$$
$$\simeq .360$$

$\underline{\xi = +i}$

$$R(\Omega'/\tfrac{4}{\pi}) \simeq .250 - .069 - .179$$
$$\simeq 0$$

$\underline{\xi = 0}$

$$R(\Omega'/\tfrac{4}{\pi}) = .507$$

$\underline{\xi = 1}$

$R(\Omega'/\frac{4}{\pi}) = \infty$

$\underline{\xi = \frac{8}{17} + \frac{15}{17}i}$ i.e. point $\mathcal{B}$

$R(\Omega'/\frac{4}{\pi}) \simeq 0$ (Note $\xi = -\alpha$ where $\alpha = \frac{8}{-17} + \frac{15}{17}i$ and thus $\xi^2, \xi^3,$ . . . have been calculated by the programme.)

SECTION 8

CONCLUSION

The re-arrangement of the equations into an acceptable form, and the choice of suitable initial values, is basically one of trial and error. No general rule can therefore be given for the choice of initial values in a particular loading case. A fairly minor increase in complexity of the shape being investigated appears to produce a large increase in the mechanical difficulties of solution. As insight into the best approach to make often depends on an overall view of the numbers involved, the difficulties in the evaluation of these numbers often prohibits more-searching analysis.

However, if there is sufficient requirement for a solution of a particular case, it seems that it could be obtained fairly satisfactorily using one or both of the approximate methods of solution of the equations. The time and effort involved could be quite large, however. Also, the availability of a computer would be, at the least, very desirable.

APPENDIX A

DESCRIPTION OF FORTRAN PROGRAMMES USED IN SECTION 4

SECTION A.1

FORTRAN PROGRAMME PROG1

This programme solves the infinite set of simultaneous equations by 'pure iteration' :

$$x_1 + x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{4}x_4 + \ldots = 100$$

$$x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \ldots = -100$$

$$x_3 + \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \ldots = 100$$

. . . .

To obtain the solutions where the r.h.s's. are ±1 instead of ±100, merely divide each x_r by 100.

The computer listing gives the values of $x_1, x_2, \ldots, x_r$ for each r_{th} iteration up to $r = 50$.

Each x_r oscillates between slowly decreasing limits and it is reasonable to assume that the mid-point of the oscillation gives a good approximation to the real solution. These mid-points are :

$x_1 = 69.3123$ $\quad$ $x_2 = -109.6611$

$x_3 = 95.3746$ $\quad$ $x_4 = -102.6601$

$x_5 = 98.3045$ $\quad$ $x_6 = -101.1531$

. . . .

$x_{23} = 100.0027$ $\quad$ $x_{24} = -99.9923$

$x_{25} = 100.0119$ $\quad$ $x_{26} = -99.9845$

. . . .

$x_{39} = 100.0333$ $\quad$ $x_{40} = -99.9664$

SECTION A.2

FORTRAN PROGRAMME PRO2

PRO2 solves the above equations with r.h.s's ±1 by first assuming

$x_r = +1$ $\quad$ r odd

$x_r = -1$ $\quad$ r even

and finding the amounts which need to be added to these values to satisfy the equations.

The first 4 lines of output are the amounts (ΔR_r) which have to be added to the l.h.s's (when the above values for the x's are assumed) to make them equal to the r.h.s's.

Thus $\quad \Delta R_1 = -\log_e 2 = -.69315$

$\Delta R_2 = -1 + \log_e 2 = -.30685$

$\Delta R_3 = \frac{1}{2} - \log_e 2 = -.019315$ $\quad$ etc.

Note that these amounts get closer to zero as the equation number increases.

The computer programme then lists

$\Delta x_1, \Delta x_2, \Delta x_3, \ldots \; x_r$ $\quad$ for each r_{th} iteration.

The first line gives these values immediately Δx_r is calculated and the secondline, the same increments after going through the iteration loop again.

(Δx_r is the amount required to be added to x_r to satisfy the equations).

'Convergence' is very fast and Δx_1 to Δx_{40} are calculated.

This gives

$x_1 = .69485 \qquad x_2 = -1.09622$

$x_3 = .95373 \qquad x_4 = -1.02675$

$x_5 = .98285 \qquad x_6 = -1.01172$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$x_{23} = 1.00000 \qquad x_{24} = -.99994$

$x_{25} = 1.00010 \qquad x_{26} = -.99986$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$x_{39} = 1.00033 \qquad x_{40} = -.99966$

Notice how close these results are to the mid-points of the x's found by PROG1 (especially for the higher x's). These results (PRO2) are of course the more accurate, wherever there is a discrepancy.

SECTION A.3

SOLUTION OF FIRST SET OF EQUATIONS

To solve

$$x_1 + x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 + \ldots = 1$$
$$x_2 + \frac{1}{2^2} x_1 + \frac{1}{3^2} x_2 + \frac{1}{4^2} x_3 + \ldots = 1$$
$$x_3 + \frac{1}{3^2} x_1 + \frac{1}{4^2} x_2 + \frac{1}{5^2} x_3 + \ldots = 1$$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

Let $x_1 = x_2 = x_3 \ldots = 1$

and call the values of the l.h.s's thus obtained $R_1, R_2, R_3, \ldots$ Call $\Delta R_1, \Delta R_2, \ldots$ the amounts that must be added to R_1, R_2, R_3, to give the values 1, 1, 1, ...

$$R_1 = 1 + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = 1 + \frac{\pi^2}{6}$$
$$R_2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = R_1 - 1$$
$$R_3 = 1 + \frac{1}{3^2} + \ldots = R_2 - \frac{1}{2^2}$$
$$R_4 = \quad . \quad . \quad . \quad = R_3 - \frac{1}{3^2} \qquad \text{etc.}$$

So $\Delta R_1 = -\frac{\pi^2}{6}$ $\qquad \Delta R_2 = -\frac{\pi^2}{6} + 1$ $\qquad$ etc.

The Fortran programme PRO3 then calculates $\Delta R_1, \ldots, \Delta R_{50}$ and thence $\Delta x_1, \ldots \Delta x_{50}$ i.e. the amounts that have to be added to the assumed values of $x_1, \ldots x_{50}$ to satisfy the equations.

Once again, 'convergence' is extremely fast, as can be seen by the listing.

Results:-

$x_1 = .25915$	$x_2 = .62360$
$x_3 = .73994$	$x_4 = .79939$
$x_5 = .83598$	$x_6 = .86905$
. .	. .
$x_{23} = .95992$	$x_{24} = .96150$
$x_{25} = .96296$	$x_{26} = .96431$
. .	. .
$x_{47} = .97973$	$x_{48} = .98014$
$x_{49} = .98053$	$x_{50} = .98091$

See the listing of PRO2 for further details.

SECTION A.4

SOLUTION OF SECOND SET OF EQUATIONS

To solve

$$x_1 + x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \ldots = 1$$

$$x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \ldots = 0$$

$$x_3 + \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \ldots = -1$$

$$x_4 + \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 + \ldots = 0$$

$$x_5 + \tfrac{1}{5}x_1 + \tfrac{1}{6}x_2 + \ . \ . \ . = 1 \qquad \text{etc.}$$

Let $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 0$, $x_5 = 1$, $x_6 = 0$, $x_7 = -1$ and so on.

Then $R_1 = 1 + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = 1 + \frac{\pi}{4} = 1.78540$

$$R_2 = 0 + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} \ldots = \tfrac{1}{2}\log_e 2 = .34657$$

$$R_3 = -1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} \ . \ . \ . = -\tfrac{\pi}{4} = -.78540$$

$$R_4 = 0 + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{8} \ . \ . \ . = \tfrac{1}{2} - \tfrac{1}{2}\log_e 2 = .15343$$

$$R_5 = 1 + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} \ . \ . \ . = R_1 - (1 - \tfrac{1}{3}) = 1.11873$$

$$R_6 = \ . \ . \ . \ . = R_2 - (\tfrac{1}{2} - \tfrac{1}{4}) = .09657 \qquad \text{etc.}$$

So $\Delta R_1 = -.78540$ $\qquad \Delta R_2 = -.34657$

$\Delta R_3 = -.21460$ $\qquad \Delta R_4 = -.15343$

$\Delta R_5 = \Delta R_1 + (1 - \frac{1}{3})$ $\qquad \Delta R_6 = \Delta R_2 + (\frac{1}{2} - \frac{1}{4})$

$\Delta R_7 = \Delta R_3 + (\frac{1}{3} - \frac{1}{5})$ $\qquad \Delta R_8 = \Delta R_4 + (\frac{1}{4} - \frac{1}{6})$

$\Delta R_9 = \Delta R_5 + (\frac{1}{5} - \frac{1}{7})$ $\qquad$ etc.

The FORTRAN programme PRO4 then calculates $\Delta R_1, \ldots \Delta R_{50}$ and thence $\Delta x_1, \ldots \Delta x_{50}$. Again 'convergence' is very fast as can be seen by the listing.

Results:-

$x_1 = .65006$	$x_2 = -.11178$
$x_3 = -1.05167$	$x_4 = -.02809$
$x_5 = .98332$	$x_6 = -.01041$
. .	. .
$x_{25} = 1.00115$	$x_{26} = .00115$
$x_{27} = -.99885$	$x_{28} = .00114$
. .	. .
$x_{47} = -.99911$	$x_{48} = .00087$
$x_{49} = 1.00086$	$x_{50} = .00085$

See the listing of PRO4 for further details.

APPENDIX B

DESCRIPTION OF FORTRAN PROGRAMME USEDIN SECTION 5(BEAM)

APPENDIX B

SECTION B.1

FORTRAN PROGRAMME BEAM2

BEAM2 first calculates ΔC_3, ΔC_5, ΔC_7, ... ΔC_{99} and these are listed on the first 4 lines of the print-out.

e.g. $\Delta C_3 = -.28320$

$\Delta C_5 = -.10027$

$\Delta C_7 = -.04960$

$\Delta C_9 = -.02911$

$\Delta C_{11} = -.01336$ etc. . .

$\Delta C_{99} = -.00124$

It then prints 2 lines for each of its calculations of

x_3, x_5, ... x_r

where r is 5 to 99.

The first line is the values of the DX's immediately after DX(R) is calculated for the first time, the second is the corresponding values after an iteration using these values finds each DX more accurately.

Having found DX(3), DX(5), DX(7), ... DX(99), x is then calculated from equation 1.

VIZ $\quad x_1 + (x_1 + \frac{1}{3}x_3 + \frac{1}{5}x_5 + \frac{1}{7}x_7 + \ldots) = -1$,

as follows: $\quad$ If $x_3 = 2$, $x_5 = -2$, $x_7 = 2$, ...

eqn. 1 is $\quad 2x_1 + 2(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} \ldots) = -1$

i.e. $\quad 2x_1 + - 1 - 2\ (1 - \frac{\pi}{4})$

But this right hand side has to be adjusted because of the values of DX(3), DX(5), ...

then $$2x_1 = -1 - 2(1 - \tfrac{\pi}{4}) - \left[\frac{DX(3)}{3} + \frac{DX(5)}{5} + \ldots \frac{DX(99)}{99}\right]$$

$$\underline{\underline{x_1 = -.67105}}$$

x_1 is printed on the last line of the computer listing.

APPENDIX C

DESCRIPTION OF FORTRAN PROGRAMMES USED IN SECTION 6

(CANTILEVER WITH END LOAD)

APPENDIX C

SECTION C.1

FORTRÁN PROGRAMME CANT ITER TO SOLVE THE EQUATIONS PRODUCED BY THE CANTILEVER BY PURE ITERATION (METHOD 1)

Refer to the listing of the values calculated by the programme (pages 2 to 4):

Lines 1 to 9 are the right hand sides of equations (0)to(99)

Lines 10 to 18 are 1, $\frac{1}{2}-1$, $\frac{3}{8}-\frac{1}{2}+1$, $\frac{15}{48}-\frac{3}{8}+\frac{1}{2}-1$, ...

Lines 19 to 27 are 1, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{15}{48}$, $\frac{105}{384}$, ...

(Note that an error in the FORTRAN compiler caused line 22 to not print. Also the characters = ( ) + are printed incorrectly as # % ' ε).

The programme then uses the above values to find the coefficients of the x_n, but these are not printed.

Line 28 lists the first iteration :

$x_1=-100/2=-50.0 \qquad x_2=-100/\left(1-\frac{2}{3}(\frac{1}{2}-1)-1.\frac{1}{2}\right)=-120.0$

Line 29 lists the second iteration:

$x_1=\left(-100+\frac{1}{2}x_2\right)/2=-80.0 \qquad x_2=-100/\left(1-\frac{2}{3}(\frac{1}{2}-1)-1.\frac{1}{2}\right)=-120.0$

$$x_3=\left(100-x_1(\tfrac{1}{2}-\tfrac{1}{2}+1)-x_2(-\tfrac{3}{4}(-\tfrac{1}{2})+\tfrac{2}{3}(-\tfrac{1}{2}))\right)/\left(1+\tfrac{3}{5}(\tfrac{3}{8}-\tfrac{1}{2}+1)+\tfrac{2}{4}(\tfrac{1}{2}-1)\right)$$
$$= 145.09807$$

and so on.

SECTION C.2

Fortran programme RESIDUAL2 to calculate residuals for cantilever equations by second method.

The first run of this programme calculates the values of the l.h.s's for 99 equations with the assumed initial values. This run considered only 40 terms of the infinite series obtained in the equations. The second run calculated for 9 equations only but considered 100 terms of each infinite series and is therefore more accurate for those 9 equations.

The programme calculates and prints:

lines 1 to 15	$1,\ \frac{1}{2},\ \frac{3}{8},\ \frac{15}{48},\ \frac{105}{384},\ \ldots$
lines 16 to 30	$1, 1-\frac{1}{2}, 1-\frac{1}{2}+\frac{3}{8}, 1-\frac{1}{2}+\frac{3}{8}-\frac{15}{48},\ \ldots$
lines 31 to 45	$1, 1-\frac{1}{2}+\frac{1}{3}, 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}, 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5},\ \ldots$
lines 46 to 60	$1, 1-\frac{1}{2\cdot3}, 1-\frac{1}{2\cdot3}+\frac{3}{8\cdot5}, 1-\frac{1}{2\cdot3}+\frac{3}{8\cdot5}-\frac{15}{48\cdot7},\ldots$

It then prints the finite parts of equations 1 to 99.

VIZ line 61	$-1+2\left(1-\sinh^{-1}1\right)-1.\frac{1}{2\cdot3}$	$= -.92943$
line 62	$(1-\frac{1}{2})-3\left(1-\sinh^{-1}1\right)$	$= .14414$
line 63	$(1-\frac{1}{2})+4\left(\sinh^{-1}1-(1-\frac{1}{2\cdot3})\right)+(1-\frac{1}{2})\frac{3}{3\cdot4\cdot5}$	$= .71719$
line 64	$-1(1-\frac{1}{2}+\frac{3}{8})-5\left(\sinh^{-1}1-(1-\frac{1}{2\cdot3})\right)$	$=-1.11523$

and so on.

It then calculates the infinite series to 40 and 100 terms respectively, taking only half of the last term in the series to take the mid-point of the oscillation.

It then adds the finite part to the infinite part and prints out the left hand sides of the 99 equations, on lines 160 to 167.

The following table shows some of the values calculated by method 1, method 2 RUN 1 and method 2 RUN 2.

EQUATION	METHOD 1	METHOD 2 RUN 1	METHOD 2 RUN 2
1	-.86969	-.86888	-.86948
2	.11614	.11536	.11594
3	.69591	.69590	.69610
4	-1.10900	-1.10959	-1.10918
5	-.72218	-.72155	-.72199
6	.36864	.36802	.36845
7	.70553	.70603	.70572
8	-.99334	-.99384	-.99354
.			
.			
.			
43	.70727	.70756	
44	-.82810	-.82839	
45	-.70754	-.70744	
46	.58907	.58897	
47	.70729	.70757	
48	-.82294	-.82322	

SECTION C.3

FORTRAN PROGRAMME CANTINF TO CALCULATE THE b_n.

CANTINF calculates the left hand sides of the equations (p.39) (0),(1),(2), . . .(101), by method 1 and divides all the l.h.s's by $\frac{1}{\sqrt{2}}$. It then prints these on lines 2 to 9. Lines 10 to 18 are then the amounts which have to be added to the original values of the l.h.s's to make them equal to the r.h.s's.

It then prints the successive iterations

$x_1 \quad x_2$

$x_1 \quad x_2 \quad x_3$

and so on.

(Note that the printout of T(1), T(2), . . . T(300)

F(1), F(2), . . . F(300)

R(1), R(2), . . . R(300)

has not been included as T(1) to T(200), F(1) to F(200) and R(1) to R(200) are shown in RESIDUAL 2 .)

It also punches the x_n on cards for input to programme CAN2.

SECTION C.4

FORTRAN PROGRAMME CAN2 WHICH READS THE b_n FROM CARDS AND CALCULATES $n a_n$ AND HENCE THE STRESSES

The programme reads the values calculated by CANTINF and multiplies each by $\frac{4}{\pi\sqrt{2}}$. This is then the values of y_1, y_2 . . . It also calculates $x_1, x_2, \ldots$ although these are not used. It prints

x_1 to x_{70} on lines 1 to 6.

and y_1 to y_{70} on lines 7 to 12.

It then calculates and prints the real part of $\bar{u}_0, \bar{\mu}_1, \bar{u}_2, \ldots u_{70}$ on lines 13 to 18,

and the imaginary part of $\bar{u}_0, \bar{u}_1, \bar{u}_2, \ldots \bar{u}_{70}$ on lines 19-24.

The next 60 lines contain, for n=1 to 60 respectively,

Real part of $n(\bar{u}_0 y_{n+1} + \bar{u}_1 y_{n+2} + \ldots \bar{u}_{69-n-1} y_{69} + \frac{1}{2}\bar{u}_{70-n-1} y_{70})$,

Real part of $n(\frac{1}{2}\bar{u}_{70-n-1} y_{70})$,

Imaginary part of $n(\bar{u}_0 y_{n+1} + \bar{u}_1 y_{n+2} + \ldots \bar{u}_{69-n-1} y_{69}$,
$+\frac{1}{2}\bar{u}_{70-n-1} y_{70})$

and imaginary part of $n(\frac{1}{2}\bar{u}_{70-n-1} y_{70})$.

The infinite sum has been approximated to by taking the middle of the oscillation at the highest term for which we had a value of y_n VIZ. y_{70}.

The absolute value of the infinite series seems to decrease quite nicely as n increases, but the absolute value of the last term we have taken unfortunately increases.

This is because n increases and also because the $\bar{u}_{70-n-1}$ is necessarily quite large because $70-n-1$ is small. Hopefully, this will not be very serious since the middle of the oscillation has been taken. Also, this last term is extremely small for low values of n, and na_n for high n is only significant on the boundary (where ξ^n still is large).

The programme then chooses various points on the diagram to calculate ξ for the corresponding Z VIZ.

$$\frac{\pi}{12}+i0,\ \frac{2\pi}{12}+i0,\ \frac{3\pi}{12}+i0,\ \ldots,\ \frac{6\pi}{12}$$

$$0+i\frac{\pi}{12},\ \frac{\pi}{12}+i\frac{\pi}{12},\ \frac{2\pi}{12}+i\frac{\pi}{12},\ \ldots,\ \frac{6\pi}{12}+i\frac{\pi}{12}$$

$$0+i\frac{2\pi}{12},\ \frac{\pi}{12}+i\frac{2\pi}{12},\ \ldots,\ \frac{6\pi}{12}+i\frac{2\pi}{12}$$

$$0+i\frac{4\pi}{12},\quad .\quad .\quad .$$

$$0+i\frac{8\pi}{12},\quad .\quad .\quad .$$

$$0+\frac{16\pi}{12}\quad .\quad .\quad .$$

$$0+\frac{32\pi}{12},\quad .\quad .\quad .$$

For each value Z it prints the real value of $\xi^1, \xi^2, \ldots \xi^{70}$ on the first 6 lines and the corresponding imaginary value on the next 6 lines. It then prints, on 2 lines,

$$R\Omega''_z, R\Omega'_z, I\Omega''_z, I\Omega'_z, R\omega''_z, I\omega''_z, x, y, \hat{x}\hat{x}+\hat{y}\hat{y},$$

$$R\left(-\tfrac{1}{2}(z\Omega''+\omega'')\right), I\left[-\left(-\tfrac{1}{2}(z\Omega''+\omega'')\right)\right], \hat{x}\hat{x}, \hat{y}\hat{y}, \hat{x}\hat{y}, R\frac{d\xi}{dz}, I\frac{d\xi}{dz}$$

APPENDIX D

SUCCESSIVE APPROXIMATION METHOD FOR LOADING CASE(1) OF CHANGE OF SECTION.

SECTION D.1

CALCULATION OF INITIAL VALUES (METHOD 1)

Although re-organization case 2 seemed far more reliable than case 1, the latter was first produced and so, in an attempt to find initial values capable of giving right hand sides close to the required ones, it was assumed

$$x_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

i.e. $\left.\begin{array}{l} \alpha_n = 1 \\ \beta n = 0 \end{array}\right\}$ all n

(this does not affect the theory derived, as the values calculated need merely be multiplied by a constant)

Substituting these values in eqns. on p. 72 gives

1. $\theta_1 + i\phi_1$

$\quad +\big((\theta_1+i\phi_1) + (\theta_2+i\phi_2) + (\theta_3+i\phi_3) + \ldots \big) = C_1$

2. $(\theta_1+i\phi_1) + 2(\theta_2+i\phi_2)$

$\quad +2\big((\theta_2+i\phi_2) + (\theta_3+i\phi_3) + \ldots \big) = C_2$

3. $(\theta_1+i\phi_1) + 2(\theta_2+i\phi_2) + 3(\theta_3+i\phi_3)$

$\quad +3\big((\theta_3+i\phi_3) + (\theta_4+i\phi_4) + (\theta_5+i\phi_5) + \big) = C_3$

4. $(\theta_1+i\phi_1) + 2(\theta_2+i\phi_2) + 3(\theta_3+i\phi_3) + 4(\theta_4+i\phi_4)$

$\quad +4\big((\theta_4+i\phi_4) + (\theta_5+i\phi_5) + (\theta_6+i\phi_6) + \ldots \big) = C_4$

To simplify the calculation, let

$$-\frac{1}{2} - \frac{1}{2}\alpha = \lambda_1$$

$$-\frac{1}{2^2\,2!} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\alpha\right) + 1\left(\frac{1\cdot 3}{2^2\,2!}\alpha^2\right) = \lambda_2$$

$$-\frac{1\cdot 3}{2^3\,3!} + \left(-\frac{1}{2^2\,2!}\right)\left(-\frac{1}{2}\alpha\right) - \frac{1}{2}\left(\frac{1\cdot 3}{2^2\,2!}\alpha^2\right) + 1.\left(-\frac{1\cdot 3\cdot 5}{2^3\,3!}\alpha^3\right) = \lambda_3$$

etc.

then $\frac{(1-\xi)^{1/2}}{(1+\alpha\xi)^{1/2}} = 1 + \lambda_1\xi + \lambda_2\xi^2 + \lambda_3\xi^3 + \ldots$

(refer to eqn. 7.2)

First, let $\xi = 1$, then the above eqn. gives

$0 = 1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \ldots$ †

And, from eqn.(7.3), p. 79

$\theta_1 + i\phi_1 = -B(1)$

$2(\theta_2 + i\phi_2) = -B(\lambda_1)$

$3(\theta_3 + i\phi_3) = -B(\lambda_2 - 1)$

$4(\theta_4 + i\phi_4) = -B(\lambda_3 - \lambda_1)$

$5(\theta_5 + i\phi_5) = -B(\lambda_4 - \lambda_2 + 1)$

$6(\theta_6 + i\phi_6) = -B(\lambda_5 - \lambda_3 + \lambda_1)$

$7(\theta_7 + i\phi_7) = -B(\lambda_6 - \lambda_4 + \lambda_2 - 1)$ etc.

So

$$-\frac{1}{B}\sum_{r=1}^{N} r(\theta_r + i\phi_r) = \begin{cases} N=4I+1 & 1+\lambda_3+\lambda_4+\lambda_7+\lambda_8+\ldots+\lambda_{4I+1}+\lambda_{4I} \\ N=4I+2 & 1+\lambda_1+\lambda_4+\lambda_5+\ldots+\lambda_{4I}+\lambda_{4I+1} \\ N=4I+3 & \lambda_1+\lambda_2+\lambda_5+\lambda_6+\ldots+\lambda_{4I+1}+\lambda_{4I+2} \\ N=4I+4 & \lambda_2+\lambda_3+\lambda_6+\lambda_7+\ldots+\lambda_{4I+2}+\lambda_{4I+3} \end{cases}$$

where I is integral. (eqn. Δ)

Now

$$C_N = \left((\theta_1+i\phi_1)+2(\theta_2+i\phi_2)+3(\theta_3+i\phi_3) \ldots +N(\theta_N+i\phi_N)\right) + N\left((\theta_N+i\phi_N)+(\theta_{N+1}+i\phi_{N+1})+(\theta_{N+2}+i\phi_{N+2})+ \ldots\right)$$

$$C_N = \left((\theta_1+i\phi_1)+2(\theta_2+i\phi_2)+3(\theta_3+i\phi_3)+ \ldots +N(\theta_N+i\phi_N)+(N+1)(\theta_{N+1}+i\phi_{N+1})+(N+2)(\theta_{N+2}+i\phi_{N+2}) + \ldots\right)$$

$$+N(\theta_N+i\phi_N)-\left(\frac{1}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1})+\frac{2}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2}) + \frac{3}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3})+\ldots+\frac{M}{N+M}(N+M)(\theta_{N+M}+i\phi_{N+M})+\ldots\right)$$

. . . . (A)

Also

$$C_{N+2} = \left((\theta_1+i\phi_1)+2(\theta_2+i\phi_2)+3(\theta_3+i\phi_3) + \ldots\right)$$
$$+(N+2)(\theta_{N+2}+i\phi_{N+2})-\left(\frac{1}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3})+\frac{2}{N+4}(N+4)\cdot\right.$$
$$(\theta_{N+4}+i\phi_{N+4})+\frac{3}{N+5}(N+5)(\theta_{N+5}+i\phi_{N+3})+ \ldots$$
$$\left.+\frac{M-2}{N+M}(N+M)(\theta_{N+M}+i\phi_{N+M}) + \ldots\right)$$

So, for all N,

$$\frac{C_N+C_{N+2}}{-B} = \left(1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+ \ldots\right)$$
$$+\left(\lambda_{N+1}\right)-\left(\frac{1}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1})+\frac{2}{N+2}(N+2)\cdot\right.$$
$$(\theta_{N+2}+i\phi_{N+2}) +\frac{1+3}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3})+\frac{2+4}{N+4}(N+4)(\theta_{N+4}+i\phi_{N+4})$$
$$+\frac{3+5}{N+5}(N+5)(\theta_{N+5}+i\phi_{N+5})+\ldots+\frac{M-2+M}{N+M}(N+M)$$
$$\left.(\theta_{N+M}+i\phi_{N+M})+ \ldots\right)$$
$$= 0+\lambda_{N+1} - S_N^{\omega} \qquad \text{from eqn.}\dagger\ (\text{P.106})$$

and letting

$$S_N^{\infty} =\frac{1}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1})+\frac{2}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2})$$
$$+\frac{1+3}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3})+\frac{2+4}{N+4}(N+4)(\theta_{N+4}+i\phi_{N+4})+ \ldots$$

Case 1 Let $N=4I$ where I integer.

$$S_N^{\infty} =\frac{1}{N+1}\left(\lambda_N-\lambda_{N-2}+\ldots-\lambda_2+1\right)+\frac{2}{N+2}\left(\lambda_{N+1}-\lambda_{N-1}+\ldots-\lambda_3+\lambda_1\right)$$
$$+\frac{1+3}{N+3}\left(\lambda_{N+2}-\lambda_N+\lambda_{N-2}\ldots+\lambda_2-1\right)+\frac{2+4}{N+4}\left(\lambda_{N+3}-\lambda_{N+1}+\lambda_{N-1}\ldots\right.$$
$$\left.+\lambda_3-\lambda_1\right)+\frac{3+5}{N+5}\left(\lambda_{N+4}-\lambda_{N+2}\ldots-\lambda_2+1\right)+\frac{4+6}{N+6}\left(\lambda_{N+5}-\lambda_{N+3}+\ldots\right.$$
$$\left.-\lambda_3+\lambda_1\right)+ \ldots +\frac{2M}{N+M+1}\left(\lambda_{N+M}-\lambda_{N+M-2} \ldots\right)$$
$$+\frac{2M+2}{N+M+2}\left(\lambda_{N+M+1}- \ldots\right) + \ldots \tag{B}$$

$$S_N^{\infty} = 1\left(\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}- \ldots\right) +\lambda_1\left(\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}- \ldots\right)$$
$$-\lambda_2\left(\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}- \ldots\right) -\lambda_3\left(\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}- \ldots\right)$$
$$+\lambda_4\left(\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}- \ldots\right) +\lambda_5\left(\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}- \ldots\right)$$
$$+ \ldots$$

$$+\lambda_{N+1}\left(\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}\quad . \quad . \quad .\right)$$
$$+\ \alpha$$

where $\alpha = \lambda_{N+2}\left(\frac{1+3}{N+3}-\frac{3+5}{N+5}+\frac{5+7}{N+7}- \quad . \quad . \quad . \right)$

$$+\lambda_{N+3}\left(\frac{2+4}{N+4}-\frac{4+6}{N+6}+\frac{6+8}{N+8}- \quad . \quad . \quad . \right)$$
$$+\lambda_{N+4}\left(\frac{3+5}{N+5}-\frac{5+7}{N+7}+\frac{7+9}{N+9}- \quad . \quad . \quad . \right)$$
$$+\lambda_{N+5}\left(\frac{4+6}{N+6}-\frac{6+8}{N+8}+\frac{8+10}{N+10}- \quad . \quad . \quad . \right)$$
$$+ \quad . \quad . \quad .$$

Let $\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}- \quad . \quad . \quad . = S_1(N)$

and $\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}- \quad . \quad . \quad . = S_2(N)$

These series do not converge but, by taking the middle of the oscillation, apparently useful results are obtained.

See APPENDIX E.

then $S_{N\infty} = S_1(N)(1-\lambda_2+\lambda_4 \quad . \quad . \quad . +\lambda_N)$

$$+S_2(N)(\lambda_1-\lambda_3+\lambda_5 \quad . \quad . \quad . +\lambda_{N+1})$$
$$+\ \alpha$$

and

$$\alpha = \lambda_{N+2}\left(\frac{1}{N+1}-S_1(N)\right)+\lambda_{N+3}\left(\frac{2}{N+2}-S_2(N)\right)$$
$$+\lambda_{N+4}\left(S_1(N)-\left\{\frac{1}{N+1}-\frac{1+3}{N+3}\right\}\right)+\lambda_{N+5}\left(S_2(N)-\left\{\frac{2}{N+2}-\frac{2+4}{N+4}\right\}\right)$$
$$+\lambda_{N+6}\left(\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}-S_1(N)\right)+\lambda_{N+7}\left(\left\{\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}\right\}-S_2(N)\right)$$
$$+ \quad . \quad . \quad .$$

$$\underline{S_{N\infty} = S_1(N).A + S_2(N).B' + \phi_N}$$

where $A = 1-\lambda_2+\lambda_4 \quad . \quad . \quad . +\lambda_N-\lambda_{N+2}+\lambda_{N+4}- \quad . \quad . \quad .$

and $B' = \lambda_1-\lambda_3+\lambda_5 \quad . \quad . \quad . +\lambda_{N+1}-\lambda_{N+3}+\lambda_{N+5}- \quad . \quad . \quad .$

See APP. E for calculation of A and B'.

and $\phi_N = \frac{1}{N+1}\lambda_{N+2}-\left(\frac{1}{N+1}-\frac{1+3}{N+3}\right)\lambda_{N+4}+\left(\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}\right)\lambda_{N+6}- \quad . \quad . \quad .$

$$+\frac{2}{N+2}\lambda_{N+3}-\left(\frac{2}{N+2}-\frac{2+4}{N+4}\right)\lambda_{N+5}+\left(\frac{2}{N+2}-\frac{2+4}{N+4}+\frac{4+6}{N+6}\right)\lambda_{N+7}-\ \cdot\ \cdot\ \cdot$$

Case 2 $\quad N=4I+2$

$$S_{N^\infty} = \frac{1}{N+1}\left(\lambda_N-\lambda_{N-2}\cdots+\lambda_2-1\right)+\frac{2}{N+2}\left(\lambda_{N+1}-\lambda_{N-1}\cdots+\lambda_3-\lambda_1\right)$$
$$+\frac{1+3}{N+3}\left(\lambda_{N+2}-\lambda_N\cdots-\lambda_2+1\right)+\frac{2+4}{N+4}\left(\lambda_{N+3}-\lambda_{N+1}\cdots-\lambda_3+\lambda_1\right)$$
$$+\frac{3+5}{N+5}\left(\lambda_{N+4}-\lambda_{N+2}\cdots+\lambda_2-1\right)+\frac{4+6}{N+6}\left(\lambda_{N+3}-\lambda_{N+1}\cdots+\lambda_3-\lambda_1\right)$$
$$+\quad\cdot\quad\cdot\quad\cdot$$

$$S_{N^\infty} = -1.S_1(N)-\lambda_1.S_2(N)+\lambda_2S_1(N)+\lambda_3S_2(N)-\lambda_4S_1(N)+\ldots$$
$$+\lambda_NS_1(N)+\lambda_{N+1}S_2(N)+\lambda_{N+2}\left(\frac{1+3}{N+3}-\frac{3+5}{N+5}\cdots\right)+$$
$$\lambda_{N+3}\left(\frac{2+4}{N+4}-\frac{4+6}{N+6}+\ldots\right)+\lambda_{N+4}\left(\frac{3+5}{N+5}-\frac{5+7}{N+7}\cdots\right)+$$
$$\lambda_{N+5}\left(\frac{4+6}{N+6}-\frac{6+8}{N+8}+\ldots\right)\ +\ \cdot\ \cdot\ \cdot$$
$$S_{N^\infty} = -S_1(N).A - S_2(N).B'+\phi_N$$

Case 3 $\quad N = 4I+1$

$$S_{N^\infty} = \frac{1}{N+1}\left(\lambda_N-\lambda_{N-2}\cdots-\lambda_3+\lambda_1\right)+\frac{2}{N+2}\left(\lambda_{N+1}-\lambda_{N-1}\cdots+\lambda_2-1\right)$$
$$+\frac{1+3}{N+3}\left(\lambda_{N+2}-\lambda_N\cdots+\lambda_3-\lambda_1\right)+\frac{2+4}{N+4}\left(\lambda_{N+3}-\cdots-\lambda_2+1\right)$$
$$+\quad\cdot\quad\cdot\quad\cdot$$
$$=-1.S_2(N)+\lambda_1S_1(N)+\lambda_2S_2(N)-\lambda_3S_1(N)-\lambda_4S_2(N)+\lambda_5S_1(N)$$
$$+\ \cdot\ \cdot\ \cdot\ +\lambda_NS_1(N)+\lambda_{N+1}S_2(N)+\phi_N$$
$$= -S_2(N).A + S_1(N).B' + \phi_N$$

Case 4 $\quad N = 4I+3$

$$S_{N^\infty} = \frac{1}{N+1}\left(\lambda_N-\lambda_{N-2}+\cdots+\lambda_3-\lambda_1\right)+\frac{2}{N+2}\left(\lambda_{N+1}-\lambda_{N-1}\cdots-\lambda_2+1\right)$$
$$+\frac{1+3}{N+3}\left(\lambda_{N+2}-\lambda_N\cdots-\lambda_3+\lambda_1\right)+\frac{2+4}{N+4}\left(\lambda_{N+1}\cdots+\lambda_2-1\right)\ \ldots$$
$$= 1.S_2(N)-\lambda_1S_1(N)-\lambda_2S_2(N)+\lambda_3S_1(N)+\ \cdot\ \cdot\ \cdot\ +\phi_N$$

$= S_2(N).A - S_1(N).B' + \phi_n$

Now $\phi_n = \frac{1}{N+1}\lambda_{N+2} - (\frac{1}{N+1} - \frac{1+3}{N+3})\lambda_{N+4} + \ldots$

$\pm (\frac{1}{N+1} - \frac{1+3}{N+3} \ldots \pm \frac{M-2+M}{N+M})\lambda_{N+M+1}$

$\mp (\frac{1}{N+1} - \ldots \pm \frac{M-2+M}{N+M} \mp \frac{M+M+2}{N+M+2})\lambda_{N+M+3} \pm \ldots$

$+ \frac{2}{N+2}\lambda_{N+3} - (\frac{2}{N+2} - \frac{2+4}{N+4})\lambda_{N+5} + \ldots$

$\pm (\frac{2}{N+2} - \frac{2+4}{N+4} \ldots \pm \frac{M-1+M+1}{N+M+1})\lambda_{N+M+2}$

$\mp (\frac{2}{N+2} - \frac{2+4}{N+4} \ldots \pm \frac{M-1+M+1}{N+M+1} \mp \frac{M+1+M+3}{N+M+3})\lambda_{N+M+4} \pm \ldots$

Take $M = 4I+1$ where I integral and $M >> N$.

Since $\frac{2M+2}{N+M+2} \simeq 2$ for $M >> N$ and $S_1(N)$ and $S_2(N) \simeq 0$ for high N

$- (\frac{1}{N+1} - \ldots - \frac{M+M+2}{N+M+2}) \simeq 1$

and $+ (\frac{1}{N+1} \ldots + \frac{M+2+M+4}{N+M+4}) \simeq 1$

[This assumes the value of the series oscillates between +1 and -1]

for high N. (This is not so for low N, but in this 'solution' it is incorrectly assumed to be the case. This could be remediedbut the programme COSI(1) was written using this assumptionand a different method which remedies other possible anomalies is used later.)

So $\phi_n \simeq \frac{1}{N+1}\lambda_{N+2} - (\frac{1}{N+1} - \frac{1+3}{N+3})\lambda_{N+4} + \ldots + (\frac{1}{N+1} - \frac{1+3}{N+3} + \ldots + \frac{M-2+M}{N+M})\lambda_{N+M+1}$

$+ \frac{2}{N+2}\lambda_{N+3} - (\frac{2}{N+2} - \frac{2+4}{N+4})\lambda_{N+5} \ldots + (\frac{2}{N+2} - \frac{2+4}{N+4} \ldots + \frac{2M}{N+M+1})\lambda_{N+M+2}$

$+ \lambda_{N+M+3} + \lambda_{N+M+4} + \lambda_{N+M+5} + \ldots$

(The coefficients of the λ_n's in this last series may differ from 1 and the inaccuracies this produces depend on the values of the λ's — small but not insignificant even at the high values of M (maximum 197) we consider)

$$\phi_n = \frac{1}{N+1}\lambda_{N+2} - \quad \cdots$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$- (1+\lambda_1+\lambda_2+\lambda_3+\ldots+ {}_{N+M+2}) \qquad\qquad \text{from eqn. } \dagger \text{ p.106}$$

And

$$\left.\begin{aligned} \frac{C_N+C_{N+2}}{-B} &= \lambda_{N+1} - S_1(N).A - S_2(N).B' - \phi_N \\ &= \lambda_{N+1} + S_1(N).A + S_2(N).B' - \phi_N \\ &= \lambda_{N+1} + S_2(N).A - S_1(N).B' - \phi_N \\ &= \lambda_{N+1} - S_2(N).A + S_1(N).B' - \phi_n \end{aligned}\right\} * \quad \begin{aligned} &N=4I \\ &N=4I+2 \\ &N=4I+1 \\ &N=4I+3 \end{aligned}$$

(Note that the terms in $S_1(N)$ and $S_2(N)$ are insignificant for high N and λ_{N+1} and ϕ_N are of approximately equal significance.)

So, adding eqn. 1 and eqn. 5

eqn. 2 and eqn. 6 etc.,

as for the simple iteration method, we can find the initial r.h.s.'s for equations (5) onwards.

Again, however, we need three other equations and so it is necessary to know $\frac{C_n}{-B}$.

If, in eqn. Δ (p.106), we let N approach infinity we obtain the first large bracketed term of $\frac{C_N}{-B}$. As the average of the 4 right hand sides is 0, this has been taken as the value as N approaches infinity but this is obviously very questionable as the l.h.s. obviously cannot be said to converge since it approaches 4 different values depending on N.

If we let $A = 1-\lambda_2+\lambda_4-\lambda_6+\dots$ as on p.108

and $B' = \lambda_1-\lambda_3+\lambda_5- \quad +\dots$

then as $N \to \infty$, obviously

$$A-B' = 1-\lambda_1-\lambda_2+\lambda_3+\lambda_4-\lambda_5-\lambda_6+\lambda_7+\lambda_8-\dots \qquad (a)$$

$$\text{but } 0 = 1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8+\dots \qquad (b)$$

$\frac{(a)+(b)}{2}$ gives $\frac{A-B'}{2} = 1+\lambda_3+\lambda_4+\lambda_7+\lambda_8+\dots$

similarly $\frac{A+B'}{2} = 1+\lambda_1+\lambda_4+\lambda_5+\lambda_8+\lambda_9+\dots$

So for N very large,

$$\left(-\frac{1}{B}\sum_{k=1}^{N} r(\theta_r+i\phi_r)\right) \simeq \left.\begin{array}{l} \frac{A-B'}{2} \\ \frac{A+B'}{2} \\ \frac{B'-A}{2} \\ -\frac{B'-A}{2} \end{array}\right\} \begin{array}{l} N=4I+1 \\ N=4I+2 \\ N=4I+3 \\ N=4I+4 \end{array}$$

However, assuming a value of 0 as N approaches infinity and using eqn. (A) on p. 106 , we get :

$$\begin{aligned} \frac{C_N}{-B} &= 0 + (\lambda_{N-1}-\lambda_{N-3}+\lambda_{N-5}- \quad \dots) \\ &\quad -\left(\frac{1}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1}) + \frac{2}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2})\right. \\ &\quad \left. + \frac{3}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3}) + \quad \dots \right) \\ &= (\lambda_{N-1}-\lambda_{N-3}+\lambda_{N-5}- \quad \dots) - S'_{N\infty} \end{aligned}$$

$$\begin{aligned} \text{where } S'_{N\infty} &= \frac{1}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1}) + \frac{2}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2}) \\ &\quad + \frac{3}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3}) + \quad \dots \end{aligned}$$

Case 1 $\quad N = 4I$

$$\begin{aligned} S'_{N\infty} &= \frac{1}{N+1}(\lambda_N-\lambda_{N-2}\dots-\lambda_2+1)+\frac{2}{N+2}(\lambda_{N+1}-\lambda_{N-1}\dots-\lambda_3+\lambda_1) \\ &\quad +\frac{3}{N+3}(\lambda_{N+2}-\lambda_N\dots+\lambda_2-1)+\frac{4}{N+4}(\lambda_{N+3}-\lambda_{N+1}+\dots+\lambda_3-\lambda_1) \\ &\quad +\frac{5}{N+5}(\lambda_{N+4}-\lambda_{N+2}+\dots-\lambda_2+1) + \quad \dots \end{aligned}$$

$$S'_{N\infty} = 1\left(\frac{1}{N+1}-\frac{3}{N+3}+\frac{5}{N+5}\quad . \quad . \quad . \right)$$
$$+\lambda_1\left(\frac{2}{N+2}-\frac{4}{N+4}+\frac{6}{N+6}\quad . \quad . \quad . \right)$$
$$-\lambda_2\left(\frac{1}{N+1}-\frac{3}{N+3}+\frac{5}{N+5}\quad . \quad . \quad . \right)$$
$$-\lambda_3\left(\frac{2}{N+2}-\frac{4}{N+4}+\frac{6}{N+6}\quad . \quad . \quad . \right)$$
$$+ \quad . \quad . \quad .$$
$$+\lambda_{N+1}\left(\frac{2}{N+2}-\frac{4}{N+4}+\frac{6}{N+6}\quad . \quad . \quad . \right)$$
$$+\ \alpha'$$

where
$$\alpha' = \lambda_{N+2}\left(\frac{3}{N+3}-\frac{5}{N+5}+\frac{7}{N+7}-\quad . \quad . \quad . \right)$$
$$+\lambda_{N+3}\left(\frac{4}{N+4}-\frac{6}{N+6}+\frac{8}{N+8}-\quad . \quad . \quad . \right)$$
$$+\lambda_{N+4}\left(\frac{5}{N+5}-\frac{7}{N+7}+\quad . \quad . \quad . \right)$$
$$+ \quad . \quad . \quad .$$

Let $\frac{1}{N+1} - \frac{3}{N+3} + \frac{5}{N+5} \quad . \quad . \quad . = S'_1(N)$

and $\frac{2}{N+2} - \frac{4}{N+4} + \frac{6}{N+6} \quad . \quad . \quad . = S'_2(N)$

See APP. D for calculation of $S'_1(N)$ and $S'_2(N)$

then $\underline{S'_{N\infty} = S'_1(N).A + S'_2(N).B' + \phi'_N}$

where
$$\phi'_N = \frac{1}{N+1}\lambda_{N+2}-\left(\frac{1}{N+1}-\frac{3}{N+3}\right)\lambda_{N+4}+\left(\frac{1}{N+1}-\frac{3}{N+3}+\frac{5}{N+5}\right)\lambda_{N+6}- \quad . \quad . \quad .$$
$$+\frac{2}{N+2}\lambda_{N+3}-\left(\frac{2}{N+2}-\frac{4}{N+4}\right)\lambda_{N+5}+\left(\frac{2}{N+2}-\frac{4}{N+4}+\frac{6}{N+6}\right)\lambda_{N+7}- \quad . \quad . \quad .$$

Case 2 $\quad N = 4I+2$

$$\underline{S'_{N\infty} = -S'_1(N).A - S'_2(N).B' + \phi'_N}$$

Case 3 $\quad N = 4I+1$

$$\underline{S'_{N\infty} = -S'_2(N).A + S'_1(N).B' + \phi'_N}$$

Case 4 $\quad N = 4I+3$

$$\underline{S'_{N\infty} = S'_2(N).A - S'_1(N).B' + \phi'_N}$$

And $\frac{C_N}{-B} = \underline{\Phi_{N-1}-S'_{N\infty}}$ where $\underline{\Phi_N = \lambda_N-\lambda_{N-2}-\lambda_{N-4}- \quad . \quad . \quad .}$

Now $\phi'_N = \frac{1}{N+1}\lambda_{N+2} - (\frac{1}{N+1} - \frac{3}{N+3})\lambda_{N+4} + \quad . \quad . \quad .$

$\pm(\frac{1}{N+1} - \frac{3}{N+3} + \ldots \pm \frac{M}{N+M})\lambda_{N+M+1}$

$\mp(\frac{1}{N+1} - \frac{3}{N+3} \quad \ldots \pm \frac{M}{N+M} \mp \frac{M+2}{N+M+2})\lambda_{N+M+3}$

$\pm \quad . \quad . \quad .$

$+\frac{2}{N+2}\lambda_{N+3} - (\frac{2}{N+2} - \frac{4}{N+4})\lambda_{N+5} + \quad . \quad . \quad .$

$\pm(\frac{2}{N+2} - \frac{4}{N+4} + \ldots \pm \frac{M+1}{N+M+1})\lambda_{N+M+2}$

$\mp(\frac{2}{N+2} - \frac{4}{N+4} + \ldots \pm \frac{M+1}{N+M+1} \mp \frac{M+3}{N+M+3})\lambda_{N+M+4}$

$\pm \quad . \quad . \quad .$

Again take $M = 4I+1$ where I integral and $M>>N$.

Since $\frac{M}{N+M} \simeq 1$ for $M>>N$

and $S'_1(N)$ and $S'_2(N)$ are very nearly 0 for high N.

$$-\left(\frac{1}{N+1} - \frac{3}{N+3} \quad . \quad . \quad . \quad - \frac{M+2}{N+M+2}\right) \simeq \frac{1}{2}$$

and $+\left(\frac{1}{N+1} \quad . \quad . \quad . \quad + \frac{M+4}{N+M+4}\right) \simeq \frac{1}{2}$ for high N.

[Again, this is not so for low N but the 'solution' when this is assumed is again followed through .]

So $\phi'_N \simeq \frac{1}{N+1}\lambda_{N+2} - (\frac{1}{N+1} - \frac{3}{N+3})\lambda_{N+4} + \ldots + (\frac{1}{N+1} - \frac{3}{N+3} \ldots + \frac{M}{N+M})\lambda_{N+M+1}$

$+\frac{2}{N+2}\lambda_{N+3} - (\frac{2}{N+2} - \frac{4}{N+4})\lambda_{N+5} \ldots + (\frac{2}{N+2} - \frac{4}{N+4} \ldots + \frac{M+1}{N+M+1})\lambda_{N+M+2}$

$-\frac{1}{2}(1 + \lambda_1 + \lambda_2 + \quad . \quad . \quad . \quad + \lambda_{N+M+2})$

Note that the term in square brackets is fairly small (but not insignificant) if N+M is sufficiently large and so the inaccuracy in the multiplier for low N should be small but not insignificant.

SECTION D.2

RESULTS OBTAINED FOR b_n.

FORTRAN programmes COSI(1) were written to find the values of the x_n using the preceding theory.

The following cases were run:

Case 1 Reorganization case 1 as for programme CALC, case 1.

i.e. Equation 1 = First equation

Equation 3 = First equation + fourth equation

Equation 4 = Third equation + fourth equation

Initial values were taken as b_n = 1 since CALC case 1 indicated these values.

Only Z(1), Z(2), and Z(3) and the iteration loops are shown.

Case 2 Reorganization Case 2 as for CALC, Case 2.

i.e. Equation 1 = Sixth equation - first equation

Equation 3 = First equation - eighth equation

Equation 4 = Third equation + fourth equation

Initial values were taken as b_n = 1. None of this programme is shown but the results are referred to - see later.

Case 2A Same reorganization case but with b_n = .53 for $N>50$ and b_n = average of last 8 iterations from COSI(1) - Case 2 for $n \leq 50$.

Z(1) to Z(50) and the iteration loops are shown.

Case 2B Same as Case 2A but with b_n = .51 for $N>50$ and b_n = average of last 8 iterations from COSI(1) - Case 2A for $n \leq 50$. The whole of this programme is shown.

Refer to APP. E (Section E.2.1) for a description of this programme.

The results of Case 1 and Case 2 again differ but from x_{12} down i.e. x_{12}, x_{13}, ... x_{50}, the results are reasonably close, becoming more so as we approach x_{50}. Remembering that 1 must be added to δx_1, δx_3, ... δx_{49} to obtain x_1, x_3, ... x_{49}, it can be seen that the results of COSI - Case 2 and CALC - Case 2 are in quite good agreement.It follows, therefore, that CALC - Case 1 and COSI - Case 1 do not agree and it does not seem worthwhile pursuing Case 1 any further. Taking the average of the last 8 oscillations (or as many oscillations as there are) for each x_n from COSI - Case 2, we obtain values for δx_n, and hence x_n (shown as 'initial x' in table below). By plotting x_n as a function of n, x_n appears to approach about .53 as $n \to \infty$.for n odd and to approach 0 as $n \to \infty$ for n even. (Note that this initial run uses an incorrect value (VIZ 1.0) for $b_n, n > 50$, and therefore is no better than the PURE ITERATION method.)

To obtain a better solution, it was then assumed that for N = 1 to 50, α_n and B_n equal the average of the last 8 values and for N>50, α_n = .53 and β_n = 0. Thus, the amounts which have to be added to the right hand sides found by assuming these initial values to give the correct r.h.s's were then found and thus the new δx_1 to δx_{50}, which are now the amounts which have to be added to the x_n in the table to give the 'more accurate' x_n. The programme used is COSI(1)

- Case 2A.

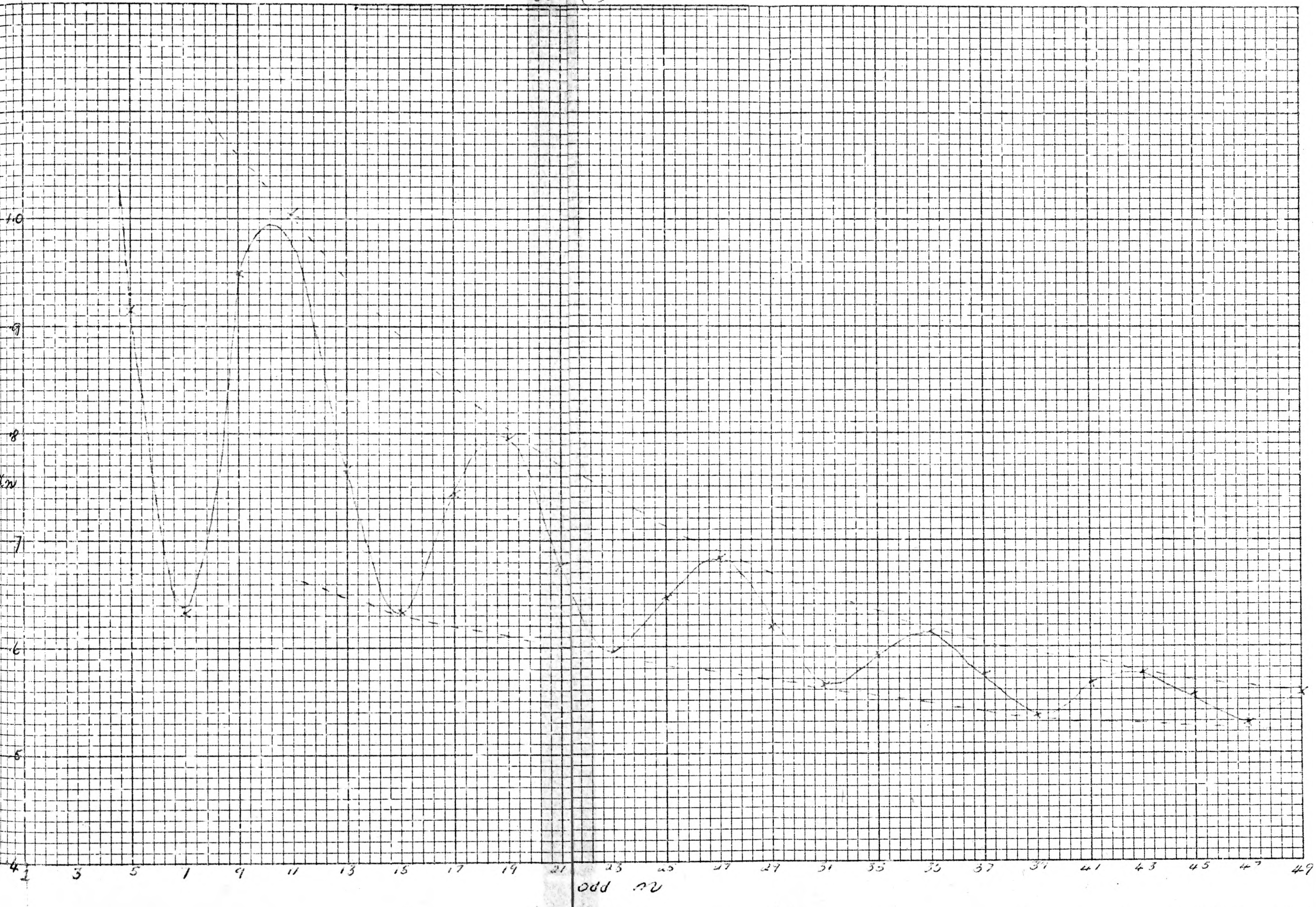
Again the average of the last 8 oscillations was found, though the variation from the average was much less in this case. Compare the table below with that for CALC Case 1.2.

	δx_n	Initial x	New x		δx_n	Initial x	New x
1	.02922	.487	.51622	2			
3	.15850	2.637	2.79550	4	.03359	.489	.52259
5	.00182	.913	.91482	6	.05488	.902	.95688
7	-.00287	.634	.63113	8	-.00733	.439	.43167
9	.06457	.885	.94957	10	-.02273	.111	.08827
11	.06607	.955	1.02107	12	.00090	.182	.18290
13	.01371	.751	.76471	14	.00984	.266	.27584
15	.00348	.631	.63448	16	.00557	.171	.17157
17	.02422	.719	.74322	18	.00531	.083	.08831
19	.02162	.773	.79462	20	.01106	.113	.12406
21	.00687	.672	.67887	22	.00845	.146	.15445
23	.00729	.588	.59529	24	.00570	.100	.10570
25	.00665	.640	.64665	26	.00456	.064	.06856
27	.00822	.673	.68122	28	.00699	.088	.09499
29	.00723	.612	.61923	30	.01112	.100	.11112
31	.00039	.564	.56439	32	.01204	.065	.07704
33	-.00343	.596	.59257	34	.00729	.046	.05329
35	.00033	.613	.61333	36	.00501	.068	.07301
37	-.00107	.575	.57393	38	.00848	.077	.08548
39	-.00806	.544	.53594	40	.00947	.053	.06247
41	-.01163	.568	.55637	42	.01037	.034	.04437
43	-.00618	.576	.56982	44	.01148	.046	.05748
45	-.01465	.556	.54135	46	.01139	.056	.06739
47	-.01630	.529	.51270	48	.01489	.036	.05089
49	-.02671	.558	.53129	50	.02070	.016	.03679

A graph of the odd x_n versus n is shown and appears as a damped oscillation approaching approximately .51.

The new values of x_n were taken to 5 decimal places as it seemed likely that the next programme run might give values with an accuracy approaching this. However COSI(1) - Case 2B did not produce smaller absolute values of δx_n for all n. This maybe due to the known false assumption on p. 110

RESULTS OF COSI(1) — CASE 2A



when N is small.

However, the results of COSI(1) - Case 2A or Case 2B should be generally accurate to a few percent which should make them useful.

SECTION D.3

CALCULATION OF INITIAL VALUES (Method 2)

As well as the one known error in the calculations for COSI(1), there are also many doubtful areas which the following alternative method hopes to eliminate. (On p.106 C_N was made into 2 infinite series instead of one finite because, when C_{N+2} was added to it the first infinite series was known to be zero VIZ $1+\lambda_1+\lambda_2+\lambda_3+ \ldots$
However this caused the second series to have coefficients which approach 1 as the terms become higher (i.e. the coefficient $\frac{M}{N+M}$ in eqn. (A) approaches 1 as $M\to\infty$).
The coefficients of the terms in eqn. (B) then approach 2 and the term in square brackets approaches plus or minus a finite constant. VIZ $\pm A$ or $\pm B'$. Thus S_N^∞ appears to oscillate. This leads to the series $S_1(N)$ and $S_2(N)$ in which the terms approach 2 and so these series also appear to oscillate. (See the calculation of these series in APP E))

Method 2 of calculating $C_N/-B$ and $(C_N+C_{N+2})/-B$

With $\alpha_N = 1$, $\beta_N = 0$

$$C_N = \left((\theta_1+i\phi_1) + 2(\theta_2+i\phi_2) + 3(\theta_3+i\phi_3) +\ldots+ N(\theta_N+i\phi_N)\right)$$
$$+ N(\theta_N+i\phi_N) + \frac{N}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1}) + \frac{N}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2})$$
$$+ \ldots \qquad --- \qquad (A')$$

Also

$$C_{N+2} = \left\{(\theta_1+i\phi_1) + 2(\theta_2+i\phi_2) + 3(\theta_3+i\phi_3) +\ldots+ (N+2)(\theta_{N+2}+i\phi_{N+2})\right)$$
$$+ (N+2)(\theta_{N+2}+i\phi_{N+2}) + \frac{N+2}{N+3}(N+3)(\theta_{N+3}+i\phi_{N+3}) + \ldots$$

So, for all N,

$$\frac{C_N+C_{N+2}}{-B} = (1 + \lambda_1 + \lambda_2 + \ldots + \lambda_{N+1}) + \lambda_{N+1} + S'(N)_\infty$$

where
$$S'(N)_\infty = \frac{N}{N+1}(N+1)(\theta_{N+1}+i\phi_{N+1}) + \frac{N}{N+2}(N+2)(\theta_{N+2}+i\phi_{N+2})$$
$$+\left(\frac{N}{N+3}+\frac{N+2}{N+3}\right)(N+3)(\theta_{N+3}+i\phi_{N+3}) + \left(\frac{N}{N+4}+\frac{N+2}{N+4}\right)(N+4)(\theta_{N+4}+i\phi_{N+4})$$
$$+\ldots$$
$$+\left(\frac{N}{N+M}+\frac{N+2}{N+M}\right)(N+M)(\theta_{N+M}+i\phi_{N+M}) + \ldots$$

Note that the coeffs. of the above terms now approach 0 as $M \to \infty$, although, unlike Method 1, they are large ($\simeq 2$) for low M.

<u>Case 1 Let N=4I</u> where I integer (with M=4J+1 J integer)

$$S'(N)_\infty = \frac{N}{N+1}(\lambda_N-\lambda_{N-2}+\ldots-\lambda_2+1) \frac{N}{N+2}(\lambda_{N+1}-\lambda_{N-1}+\ldots-\lambda_3+\lambda_1)$$
$$+ \frac{2N+2}{N+3}(\lambda_{N+2}-\lambda_N+\ldots+\lambda_2-1) + \frac{2N+2}{N+4}(\lambda_{N+3}-\lambda_{N+1}\ldots+\lambda_3-\lambda_1)$$
$$+ \frac{2N+2}{N+5}(\lambda_{N+4}-\lambda_{N+2}+\ldots-\lambda_2+1) + \frac{2N+2}{N+6}(\lambda_{N+5}-\lambda_{N+3}\ldots-\lambda_3+\lambda_1)$$
$$+\ldots+ \frac{2N+2}{N+M}(\lambda_{N+M-1}-\lambda_{N+M-3}\ldots-\lambda_2+1)$$
$$+ \frac{2N+2}{N+M+1}(\lambda_{N+M}-\lambda_{N+M-2}+\ldots-\lambda_3+\lambda_1) + \ldots \qquad \text{(B)}$$

$$S'(N)_\infty = 1.\left[\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}-\ldots\right.\right]$$
$$+\lambda_1\left[\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}-\ldots\right.\right]$$
$$-\lambda_2\left[\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}-\ldots\right.\right]$$
$$-\lambda_3\left[\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}-\ldots\right.\right]$$

$$\cdot \quad \cdot \quad \cdot$$

$$+\lambda_{N+1}\left[\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\ldots\right.\right]$$
$$+\lambda_{N+2}\left[(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\ldots\right.\right]$$
$$+\lambda_{N+3}\left[(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\ldots\right.\right]$$
$$+\lambda_{N+4}\left[(2N+2)\left(\frac{1}{N+5}-\frac{1}{N+7}+\ldots\right.\right]$$
$$+ \quad \cdot \quad \cdot \quad \cdot$$

$\cdot$

Let $T_1(N) = \frac{N}{N+1} - (2N+2)\left(\frac{1}{N+3} - \frac{1}{N+5} + \frac{1}{N+7} - \dots\right)$

$T_2(N) = \frac{N}{N+2} - (2N+2)\left(\frac{1}{N+4} - \frac{1}{N+6} + \frac{1}{N+8} - \dots\right)$

Note that $T_1(N)$ and $T_2(N)$ are found to be extremely small (see App. E and programme COSI(2)).

Then
$$
\begin{aligned}
S'(N)_\infty = {} & T_1(N)(1-\lambda_2+\lambda_4-\dots+\lambda_N) \\
& + T_2(N)(\lambda_1-\lambda_3+\dots+\lambda_{N+1}) \\
& + \lambda_{N+2}\left(\frac{N}{N+1} - T_1(N)\right) \\
& + \lambda_{N+3}\left(\frac{N}{N+2} - T_2(N)\right) \\
& + \lambda_{N+4}\left(T_1(N) - \frac{N}{N+1} + (2N+2)\frac{1}{N+3}\right) \\
& + \lambda_{N+5}\left(T_2(N) - \frac{N}{N+2} + (2N+2)\frac{1}{N+4}\right) \\
& + \lambda_{N+6}\left(\frac{N}{N+1} - (2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}\right) - T_1(N)\right) \\
& + \lambda_{N+7}\left(\frac{N}{N+2} - (2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}\right) - T_2(N)\right) \\
& + \lambda_{N+8}\left(T_1(N) - \frac{N}{N+1} + (2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}\right)\right) \\
& + \lambda_{N+9}\left(T_2(N) - \frac{N}{N+2} + (2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}\right)\right) \\
& + \dots
\end{aligned}
$$

$$
\begin{aligned}
S'(N)_\infty = {} & T_1(N).(1-\lambda_2+\lambda_4-\dots+\lambda_N-\lambda_{N+2}+\lambda_{N+4}-\dots) \\
& + T_2(N).(\lambda_1-\lambda_3+\lambda_5-\dots+\lambda_{N+1}-\lambda_{N+3}+\lambda_{N+5}-\dots) \\
& + \phi(N)
\end{aligned}
$$

$$\underline{\underline{S'(N)_\infty = T_1(N).A + T_2(N).B' + \phi(N)}}$$

where
$$
\begin{aligned}
\phi(N) = {} & \frac{N}{N+1}\lambda_{N+2} - \left(\frac{N}{N+1} - (2N+2)\left(\frac{1}{N+3}\right)\right)\lambda_{N+4} \\
& + \left(\frac{N}{N+1} - (2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}\right)\right)\lambda_{N+6} - \dots \\
& + \left(\frac{N}{N+1} - (2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\dots-\frac{1}{N+M}\right)\right)\lambda_{N+M+1}
\end{aligned}
$$

$+\frac{N}{N+2}\lambda_{N+3} - \left[\frac{N}{N+2} - (2N+2)\frac{1}{N+4}\right]\lambda_{N+5}$

$$+\left(\frac{N}{N+2} - (2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}\right)\right)\lambda_{N+7} - \dots$$

$$+ \left(\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\cdots-\frac{1}{N+M+1}\right)\right)\lambda_{N+M+2} \quad + P(N,M)_\infty$$

where $\quad M = 4J+1 \quad$ and

$$P(N,M)_\infty = -\left(\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}-\cdots+\frac{1}{N+M+2}\right)\right)\lambda_{N+M+3}$$
$$+\left(\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\cdots+\frac{1}{N+M+2}-\frac{1}{N+M+4}\right)\right)\lambda_{N+M+5}$$
$$- \quad \cdot \quad \cdot \quad \cdot$$
$$-\left(\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\cdots+\frac{1}{N+M+3}\right)\right)\lambda_{N+M+4}$$
$$+\left(\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\cdots+\frac{1}{N+M+3}-\frac{1}{N+M+5}\right)\right)\lambda_{N+M+6}$$
$$- \quad \cdot \quad \cdot \quad \cdot$$

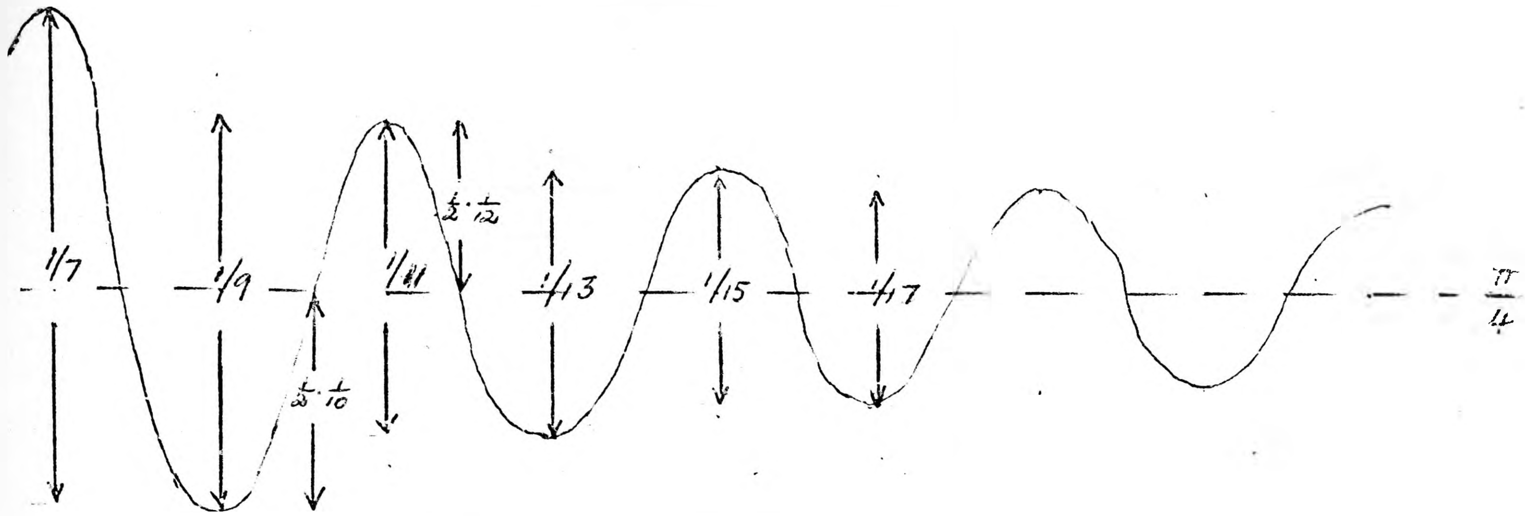
At this point we have diverged from the first method to find a more accurate value for $\phi(N)$.

$$P(N,M)_\infty \simeq -\left(\frac{N}{N+1}-(2N+2)\left(S'(N)+\frac{1}{2}\,\frac{1}{N+M+3}\right)\right)\lambda_{N+M+3}$$
$$+\left(\frac{N}{N+1}-(2N+2)\left(S'(N)-\frac{1}{2}\,\frac{1}{N+M+5}\right)\right)\lambda_{N+M+5}$$
$$-\left(\frac{N}{N+1}-(2N+2)\left(S'(N)+\frac{1}{2}\,\frac{1}{N+M+7}\right)\right)\lambda_{N+M+7}$$
$$+ \quad \cdot \quad \cdot \quad \cdot$$
$$-\left(\frac{N}{N+2}-(2N+2)\left(S'(N+1)+\frac{1}{2}\,\frac{1}{N+M+4}\right)\right)\lambda_{N+M+4}$$
$$+\left(\frac{N}{N+2}-(2N+2)\left(S'(N+1)-\frac{1}{2}\,\frac{1}{N+M+6}\right)\right)\lambda_{N+M+6}$$
$$-\left(\frac{N}{N+2}-(2N+2)\left(S'(N+1)+\frac{1}{2}\,\frac{1}{N+M+8}\right)\right)\lambda_{N+M+8} + \cdots$$

where, with $N = 4I$ where I integer

$$\begin{cases} S'(N) = \left(1-\frac{1}{3}+\frac{1}{5}\cdots+\frac{1}{N+1}\right) - \frac{\pi}{4} \\ S'(N+1) = \left(\frac{1}{2}-\frac{1}{4}+\frac{1}{6}\cdots+\frac{1}{N+2}\right) - \frac{1}{2}\log_e 2 \\ S'(N+2) = \frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\cdots-\frac{1}{N+3}\right) \\ S'(N+3) = \frac{1}{2}\log_e 2 - \left(\frac{1}{2}-\frac{1}{4}+\cdots+\frac{1}{N+4}\right) \end{cases}$$

(The approximation is very good, even for low M.



$$\text{e.g.}\quad 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7} = .72381$$

$$\text{and}\quad \tfrac{\pi}{4}-\tfrac{1}{2}\cdot\tfrac{1}{8} = .72290$$

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9} = .83492$$

$$\text{and}\quad \tfrac{\pi}{4}+\tfrac{1}{2}\cdot\tfrac{1}{10} = .83540 \quad)$$

$$\begin{aligned}
\text{i.e. } P(N,M)\infty\simeq\ & -\left(\frac{N}{N+1}-(2N+2)S'(N)\right).\left(\lambda_{N+M+3}-\lambda_{N+M+5}+\lambda_{N+M+7}-\cdots\right)\\
& -\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right).\left(\lambda_{N+M+4}-\lambda_{N+M+6}+\lambda_{N+M+8}+\cdots\right)\\
& +\left(\frac{2N+2}{2}\right)\left(\frac{1}{N+M+3}\ \lambda_{N+M+3}+\frac{1}{N+M+5}\ \lambda_{N+M+5}+\cdots\right)\\
& +\left(\frac{2N+2}{2}\right)\left(\frac{1}{N+M+4}\ \lambda_{N+M+4}+\frac{1}{N+M+6}\ \lambda_{N+M+6}+\cdots\right)\\
\simeq & -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left(\lambda_{N+M+3}-\lambda_{N+M+5}+\cdots\right)\\
& -\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left(\lambda_{N+M+4}-\lambda_{N+M+6}+\cdots\right)\\
& +(N+1)\left(\frac{\lambda_{N+M+3}}{N+M+3}+\frac{\lambda_{N+M+4}}{N+M+4}+\frac{\lambda_{N+M+5}}{N+M+5}+\cdots\right) \qquad *
\end{aligned}$$

$$\begin{aligned}
P(N,M)\infty\simeq\ & -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left(\lambda_{N+M+3}-\lambda_{N+M+5}+\lambda_{N+M+7}-\cdots\right)\\
& -\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left(\lambda_{N+M+4}-\lambda_{N+M+6}+\lambda_{N+M+8}-\cdots\right)\\
& +(N+1)\left(\left\{\frac{\lambda_{N+M+3}}{N+M+3}+\frac{\lambda_{N+M+4}}{N+M+4}+\ \cdot\ \cdot\ \cdot\ \frac{\lambda_{N+M+R}}{N+M+R}\right\}\right.\\
& \quad\left.+\left\{\frac{\lambda_{N+M+R+1}}{N+M+R}+\frac{\lambda_{N+M+R+2}}{N+M+R+1}+\frac{\lambda_{N+M+R+3}}{N+M+R+2}+\cdots\right\}\right)
\end{aligned}$$

(NOTE:- This last approximation has been made because I can find $\sum_{R=N}^{\infty}\frac{\lambda_{R+1}}{R}$ analytically (See APP. E) whereas I have not found $\sum_{R=N}^{\infty}\frac{\lambda_R}{R}$)

Case 2 $\underline{N = 4I+1}$

$$
\begin{aligned}
S'(N)_\infty = & -1\left(\frac{1}{N+2}-(2N+2)\left\{\frac{1}{N+4}-\frac{1}{N+6}+\ldots\right\}\right) \\
& +\lambda_1\left(\frac{N}{N+1}-(2N+2)\left\{\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}-\ldots\right\}\right) \\
& +\lambda_2\left(\frac{N}{N+2}-(2N+2)\left\{\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}-\ldots\right\}\right) \\
& -\lambda_3\left(\frac{N}{N+1}-(2N+2)\left\{\frac{1}{N+3}-\frac{1}{N+5}+\frac{1}{N+7}-\ldots\right\}\right) \\
& -\lambda_4\left(\frac{N}{N+2}-(2N+2)\left\{\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}-\ldots\right\}\right) \\
& + \quad \cdot \quad \cdot \quad \cdot \\
& +\lambda_{N+1}\left(\frac{N}{N+2}-(2N+2)\left\{\frac{1}{N+4}-\frac{1}{N+6}+\ldots\right\}\right) \\
& +\lambda_{N+2}\left((2N+2)\left\{\frac{1}{N+3}-\frac{1}{N+5}+\ldots\right\}\right) \\
& +\lambda_{N+3}\left((2N+2)\left\{\frac{1}{N+4}-\frac{1}{N+6}+\ldots\right\}\right) + \ldots \\
= & \; T1(N)\;(\lambda_1-\lambda_3+\lambda_5-\ldots+\lambda_N) \\
& +T2(N)\;(-1+\lambda_2-\lambda_4\ldots+\lambda_{N+1}) \\
& +\lambda_{N+2}\left(\frac{N}{N+1} - T1(N)\right) \\
& + \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

$$S'(N)_\infty = T1(N).B' - T2(N).A + \phi(N)$$

Case 3 $\underline{N = 4I+2}$

$$S'(N)_\infty = -T1(N).A - T2(N).B' + \phi(N)$$

Case 4 $\underline{N = 4I+3}$

$$S'(N)_\infty = -T1(N).B' + T2(N).A + \phi(N)$$

To calculate $\phi(N)$ take $M = 29$

$$
\begin{aligned}
\phi(N) = & \frac{N}{N+1}\;\lambda_{N+2}-\left(\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}\right)\right)\lambda_{N+4} \\
& +\left(\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}\right)\right)\lambda_{N+6}-\ldots
\end{aligned}
$$

$$+\left(\frac{N}{N+1}-(2N+2)\left(\frac{1}{N+3}-\frac{1}{N+5}+\dots-\frac{1}{N+29}\right)\right)\lambda_{N+30}$$

$$+\frac{N}{N+2}\lambda_{N+3}-\left(\frac{N}{N+2}-(2N+2)\frac{1}{N+4}\right)\lambda_{N+5}$$

$$+\left(\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}\right)\right)\lambda_{N+7}-\dots$$

$$+\left(\frac{N}{N+2}-(2N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\dots-\frac{1}{N+30}\right)\right)\lambda_{N+31}\qquad+P(N,29)_\infty$$

and (refer eqn. ✱) $P(N,29)_\infty = -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left(\lambda_{N+32}-\lambda_{N+34}\right.$

$$\left.+\dots\right)-\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left(\lambda_{N+33}-\lambda_{N+35}+\lambda_{N+37}-\dots\right)$$

$$+\Delta(N,29)$$

where $\Delta(N,29) = (N+1)\left(\frac{\lambda_{N+32}}{N+32}+\frac{\lambda_{N+33}}{N+33}+\dots\qquad\right)$

For $\underline{N=4I}$ where I integer

$$P(N,29)_\infty = -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left(A-(1-\lambda_2+\lambda_4-\dots-\lambda_{N+30})\right)$$

$$-\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left(B'-(\lambda_1-\lambda_3+\dots-\lambda_{N+31})\right)+\Delta(N,29)$$

$\underline{N=4I+1}$

$$P(N,29)_\infty = -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left(B'-(\lambda_1-\lambda_3+\dots-\lambda_{N+30})\right)$$

$$-\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left((1-\lambda_2+\dots+\lambda_{N+31})-A\right)+\Delta(N,29)$$

$\underline{N=4I+2}$

$$P(N,29)_\infty = -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left((1-\lambda_2+\dots+\lambda_{N+30})-A\right)$$

$$-\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left((\lambda_1-\lambda_3+\dots+\lambda_{N+31})-B'\right)+\Delta(N,29)$$

$\underline{N=4I+3}$

$$P(N,29)_\infty = -\left(\frac{N}{N+1}-(2N+2)S'(N)\right)\left((\lambda_1-\lambda_3+\dots+\lambda_{N+30})-B'\right)$$

$$-\left(\frac{N}{N+2}-(2N+2)S'(N+1)\right)\left(A-(1-\lambda_2+\lambda_4-\dots-\lambda_{N+31})\right)+\Delta(N,29)$$

and $\Delta(n,29) = (N+1)\left(\frac{\lambda_{N+32}}{N+32}+\frac{\lambda_{N+33}}{N+33}+\dots\quad\right)$

$$\simeq (N+1)\left[\left(\frac{\lambda_{N+32}}{N+32}+\frac{\lambda_{N+33}}{N+33}+\dots+\frac{\lambda_{N+29+R}}{N+29+R}\right)\right]$$

$$+ \left(\frac{\lambda_{N+R+30}}{N+R+29} + \frac{\lambda_{N+R+31}}{N+R+30} + \frac{\lambda_{N+R+32}}{N+R+31} + \ldots\right)$$

Take R =100

$$\Lambda(N,29) \simeq (N+1)\Big[\left(\frac{\lambda_{N+32}}{N+32} + \frac{\lambda_{N+33}}{N+33} + \ldots + \frac{\lambda_{N+129}}{N+129}\right)$$
$$+ \left(\frac{\lambda_{N+130}}{N+129} + \frac{\lambda_{N+131}}{N+130} + \frac{\lambda_{N+132}}{N+131} + \ldots\right)\Big]$$

and $\frac{C_N + C_{N+2}}{-B} = (1+\lambda_1+\lambda_2+\ldots+\lambda_{N+1}) + \lambda_{N+1}$

$$+ T_1(N).A + T_2(N).B' + \phi(N) \qquad N=4I$$
$$= (1+\lambda_1+\lambda_2+\ldots+\lambda_{N+1})+\lambda_{N+1}+T_1(N).B'-T_2(N).A+\phi(N) \qquad N=4I+1$$
$$= (1+\lambda_1+\lambda_2+\ldots+\lambda_{N+1})+\lambda_{N+1}+T_1(N).A-T_2(N).B'+\phi(N) \qquad N=4I+2$$
$$=(1+\lambda_1+\lambda_2+\ldots+\lambda_{N+1})\ \lambda_{N+1}-T_1(N).B'+T_2(N).A+\phi(N) \qquad N=4I+3$$

Now we need to find $C_N/-B$

Case 1 $N=4I$

$$C_N/-B = \lambda_2+\lambda_3+\lambda_6+\lambda_7+\ldots+\lambda_{N-2}+\lambda_{N-1}$$
$$+(\lambda_{N-1}-\lambda_{N-3}+\ldots+\lambda_3-\lambda_1)$$
$$+\frac{N}{N+1}(\lambda_N-\lambda_{N-2}+\ldots-\lambda_2+1)$$
$$+\frac{N}{N+2}(\lambda_{N+1}-\lambda_{N-1}+\ldots-\lambda_3+\lambda_1)$$
$$+\frac{N}{N+3}(\lambda_{N+2}-\lambda_N+\ldots+\lambda_2-1)$$
$$+ \ldots$$
$$= (\lambda_2 + \lambda_3 + \lambda_6 + \lambda_7 + \ldots + \lambda_{N-2} + \lambda_{N-1})$$
$$+ (\lambda_{N-1} - \lambda_{N-3} + \ldots + \lambda_3 - \lambda_1)$$
$$+ S''(N)_\infty$$

where $S''(N)_\infty = 1.N.\left(\frac{1}{N+1} - \frac{1}{N+3} + \frac{1}{N+5} - \ldots\right)$

$$+ \lambda_1.N.\left(\frac{1}{N+2} - \frac{1}{N+4} + \frac{1}{N+6} - \ldots\right)$$
$$- \lambda_2.N.\left(\frac{1}{N+1} - \frac{1}{N+3} + \frac{1}{N+5} - \ldots\right)$$

$$+ \ \ldots$$

$$+ \lambda_{N+1}.N.\left(\frac{1}{N+2} - \frac{1}{N+4} + \frac{1}{N+6} - \ldots\right)$$

$$, \quad + \lambda_{N+2}.N.\left(\frac{1}{N+3} - \frac{1}{N+5} + \frac{1}{N+7} - \ldots\right)$$

$$+ \lambda_{N+3}.N.\left(\frac{1}{N+4} - \frac{1}{N+6} + \frac{1}{N+8} - \ldots\right)$$

$$+ \ \ldots$$

Let $U_1(N) = \frac{1}{N+1} - \frac{1}{N+3} + \frac{1}{N+5} - \ldots$

$U_2(N) = \frac{1}{N+2} - \frac{1}{N+4} + \frac{1}{N+6} - \ldots$

Then $S''(N)_\infty = U_1(N).N.(1-\lambda_2+\ldots+\lambda_N) + U_2(N).N.(\lambda_1-\lambda_3+\ldots+\lambda_{N+1})$

$$+ \lambda_{N+2}.N.\left(\frac{1}{N+1}-U_1(N)\right) + \lambda_{N+4}.N.\left(U_1(N)-\left(\frac{1}{N+1}-\frac{1}{N+3}\right)\right)$$

$$+ \lambda_{N+6}.N.\left(\left(\frac{1}{N+1}-\frac{1}{N+3}+\frac{1}{N+5}\right) - U_1(N)\right) + \ldots$$

$$+ \lambda_{N+3}.N.\left(\frac{1}{N+2}-U_2(N)\right) + \lambda_{N+5}.N.\left(U_2(N)-\left(\frac{1}{N+2}-\frac{1}{N+4}\right)\right)$$

$$+ \lambda_{N+7}.N.\left(\left(\frac{1}{N+2}-\frac{1}{N+4}+\frac{1}{N+6}\right)-U_2(N)\right) + \ldots$$

$$= U_1(N).N.(1-\lambda_2+\ldots+\lambda_N-\lambda_{N+2}+\ldots)$$

$$+ U_2(N).N.(\lambda_1-\lambda_3+\ldots+\lambda_{N+1}-\lambda_{N+3}+\ldots)$$

$$+\frac{N}{N+1}\lambda_{N+2} - N\left(\frac{1}{N+1}-\frac{1}{N+3}\right)\lambda_{N+4} + N\left(\frac{1}{N+1}-\frac{1}{N+3}+\frac{1}{N+5}\right)\lambda_{N+6} - \ldots$$

$$+\frac{N}{N+2}\lambda_{N+3} - N\left(\frac{1}{N+2}-\frac{1}{N+4}\right)\lambda_{N+5} + N\left(\frac{1}{N+2}-\frac{1}{N+4}+\frac{1}{N+6}\right)\lambda_{N+7} - \ldots$$

$$\underline{\underline{S''(N)_\infty = U_1(N).N.A + U_2(N).N.B' + \phi'(N)}}$$

where $\phi'(N) = \frac{N}{N+1}\lambda_{N+2}-N\left(\frac{1}{N+1}-\frac{1}{N+3}\right)\lambda_{N+4}+N\left(\frac{1}{N+1}-\frac{1}{N+3}+\frac{1}{N+5}\right)\lambda_{N+6}-\ldots$

$$+N\left(\frac{1}{N+1}-\frac{1}{N+3}+\ldots+\frac{1}{N+M}\right)\lambda_{N+M+1}$$

$$+\frac{N}{N+2}\lambda_{N+3}-N\left(\frac{1}{N+2}-\frac{1}{N+4}\right)\lambda_{N+5}+N\left(\frac{1}{N+2}-\frac{1}{N+4}+\frac{1}{N+6}\right)\lambda_{N+7}-\ldots$$

$$+N\left(\frac{1}{N+2}-\frac{1}{N+4}+\ldots+\frac{1}{N+M+1}\right)\lambda_{N+M+2}$$

$$+P'(N,M)_\infty$$

where $M = 4J+1$ where J integer

and $P'(N,M)_\infty = -N\left(\frac{1}{N+1}-\frac{1}{N+3}+\ldots-\frac{1}{N+M+2}\right)\lambda_{N+M+3}$

$$+N\left(\frac{1}{N+1}-\frac{1}{N+3}+\ldots+\frac{1}{N+M+4}\right)\lambda_{N+M+5}$$

$$- \; . \; . \; . \; - N\left(\frac{1}{N+2}-\frac{1}{N+4}+\cdots-\frac{1}{N+M+3}\right)\lambda_{N+M+4}$$

$$+ N\left(\frac{1}{N+2}-\frac{1}{N+4}+\cdots+\frac{1}{N+M+5}\right)\lambda_{N+M+6}$$

$$- \; . \; . \; .$$

$$\begin{aligned}
P'(N.M)_\infty \simeq & -N\left(\frac{1}{N+1}-\left(S'(N)+\frac{1}{2}\;\frac{1}{N+M+3}\right)\right)\lambda_{N+M+3} \\
& +N\left(\frac{1}{N+1}-\left(S'(N)-\frac{1}{2}\;\frac{1}{N+M+5}\right)\right)\lambda_{N+M+5} \\
& -N\left(\frac{1}{N+1}-\left(S'(N)+\frac{1}{2}\;\frac{1}{N+M+7}\right)\right)\lambda_{N+M+7} \\
& + \quad . \quad . \quad . \\
& -N\left(\frac{1}{N+2}-\left(S'(N+1)+\frac{1}{2}\;\frac{1}{N+M+4}\right)\right)\lambda_{N+M+4} \\
& +N\left(\frac{1}{N+2}-\left(S'(N+1)-\frac{1}{2}\;\frac{1}{N+M+6}\right)\right)\lambda_{N+M+6} \\
& -N\left(\frac{1}{N+2}-\left(S'(N+1)+\frac{1}{2}\;\frac{1}{N+M+8}\right)\right)\lambda_{N+M+8} \\
& + \quad . \quad . \quad . \\
= & -N\left(\frac{1}{N+1}-S'(N)\right)\left(\lambda_{N+M+3}-\lambda_{N+M+5}+\cdots\right) \\
& -N\left(\frac{1}{N+2}-S'(N+1)\right)\left(\lambda_{N+M+4}-\lambda_{N+M+6}+\cdots\right) \\
& +\frac{N}{2}\left(\frac{1}{N+M+3}\lambda_{N+M+3}+\frac{1}{N+M+4}\lambda_{N+M+4}+\cdots\right)
\end{aligned}$$

The last series is the same as in eqn. * p. 124

Also when $N = 4I+1$

$$S''(N)_\infty = U_1(N).N.B' - U_2(N).N.A + \phi'(N)$$

$N = 4I+2$

$$S''(N)_\infty = -U_1(N).N.A - U_2(N).N.B' + \phi'(N)$$

$N = 4I+3$

$$S''(N)_\infty = -U_1(N).N.B' + U_2(N).N.A + \phi'(N)$$

Thus for relevant C_N we obtain

$$\frac{C_1}{-B} = 2 + U_1(1).B' - U_2(1).A + \phi'(1)$$

$$\frac{C_2}{-B} = 1+2\lambda_1-2U_1(2).A-2U_2(2).B' + \phi'(2)$$

$$\frac{C_3}{-B} = \lambda_1 - 1 - 3U_1(3).B' + 3U_2(3).A + 2\lambda_2 + \phi'(3)$$

$$\frac{C_4}{-B} = \lambda_2 + 2\lambda_3 - \lambda_1 + 4U_1(4).A + 4U_2(4).B' + \phi'(4)$$

SECTION D.4

RESULTS OBTAINED FOR b_n.

FORTRAN programmes COSI(2) found the x_n using the preceding theory and Case 2, Case 2A and Case 2B were run as for COSI(1).

The whole of Case 2A except the unadjusted and adjusted coefficients is shown and the Z(1) to Z(108) and the iteration loops of Case 2B are shown.

See APP. E (Section E.2.2) for a description of COSI(2)-Case 2A.

The values of the right hand sides calculated agree fairly closely with those of COSI(1) (within a few percent in most cases) which indicates that the theory used in COSI(1) was correct except for the one known error.

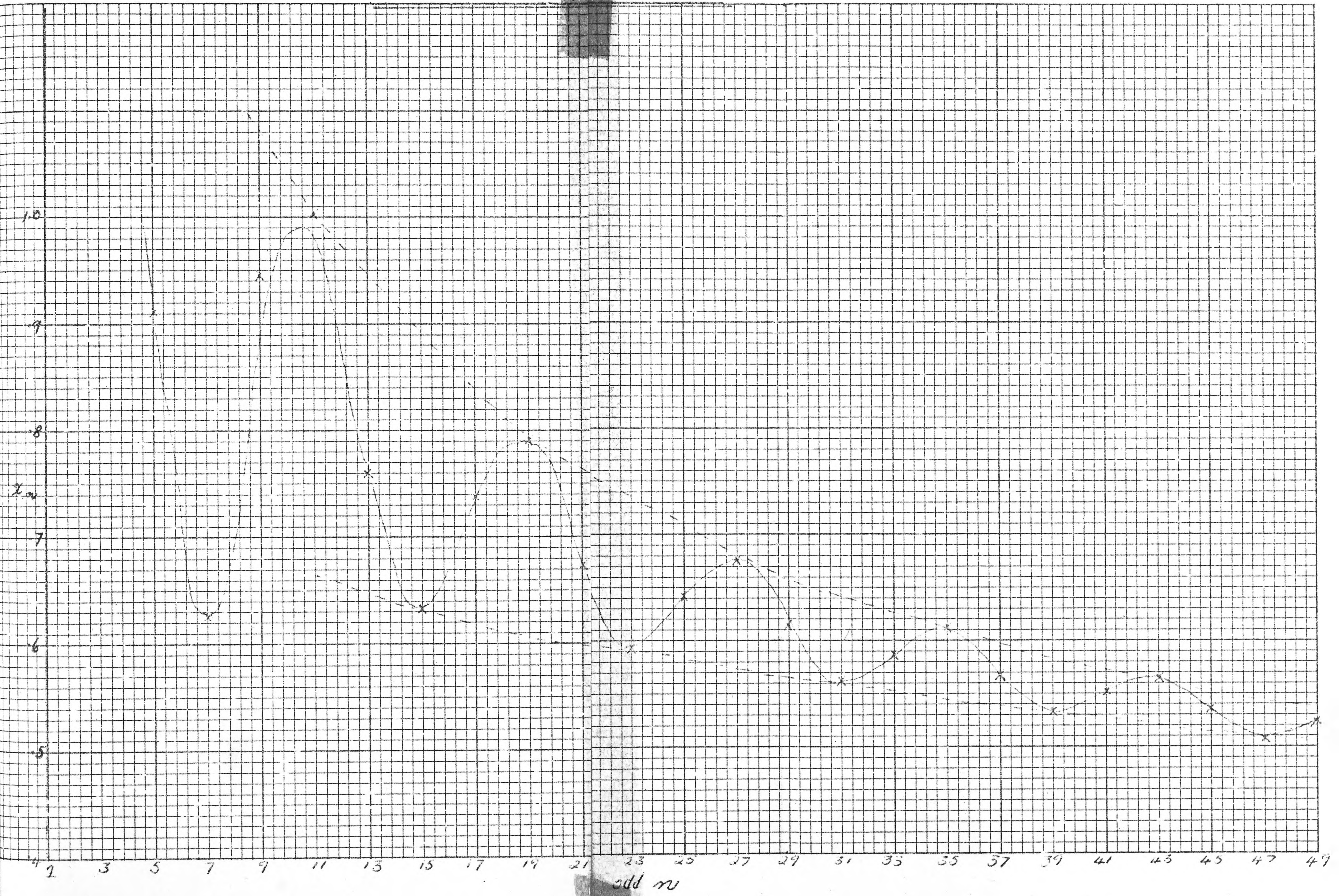
Since the coefficients of the equations are the same and the Z's are close in both COSI(1) and COSI(2), the solutions for x_n are naturally close, though the COSI(2) solutions might be excepted to be better. The equivalent table to that on p.117 is shown following.

	δx_n	Initial x	New x		δx_n	Initial x	New x
1	.02400	.48310	.50710	2			
3	.15924	2.63460	2.79384	4	.02355	.49710	.52065
5	.01176	.89920	.91096	6	.05560	.89570	.95130
7	.00224	.62260	.62484	8	.00205	.42920	.43125
9	.07552	.87910	.95462	10	-.02116	.11010	.08894
11	.03611	.98120	1.01731	12	.00168	.18110	.18278
13	.01626	.74390	.76016	14	.00827	.26690	.27517
15	.00616	.62470	.63086	16	.00339	.17370	.17709
17	.02669	.71270	.73939	18	.00399	.08550	.08949
19	.02279	.76870	.79149	20	.01017	.11450	.12467
21	.01066	.66360	.67426	22	.00829	.14650	.15479
23	.00868	.58290	.59158	24	.00378	.10310	.10688
25	.00876	.63330	.64206	26	.00532	.06540	.07072
27	.01003	.66910	.67913	28	.00551	.09140	.09691
29	.00993	.60470	.61463	30	.00821	.10460	.11281
31	.00162	.55930	.56092	32	.00952	.06980	.07932
33	-.00038	.58800	.58762	34	.00548	.05100	.05648
35	.00762	.60440	.61202	36	.00237	.07390	.07627
37	.00166	.56720	.56886	38	.00488	.08410	.08898
39	-.00586	.53870	.53284	40	.00562	.06070	.06632
41	-.00787	.55910	.55123	42	.00547	.04360	.04907
43	-.00558	.57190	.56632	44	.00799	.05430	.06229
45	-.01123	.54670	.53547	46	.00729	.06500	.07229
47	-.01513	.52410	.50897	48	.01143	.04490	.05633
49	-.02435	.54910	.52475	50	.01659	.02670	.04329

Results of COSI(2) Case 2A are similar to COSI(1) Case 2A and the new x_n are very similar in both cases. Naturally, the Case 2A results are significantly different from the Case 2 results, since completely wrong initial values were chosen for Case 2.

The results of COSI(2) Case 2B were again not as good as was hoped and were not significantly better than Case 2A. Whether this is due to the approximations assumed or an error is not known. See the computer listing for the values.

RESULTS OF COSI(2) — CASE 2A



APPENDIX E

APPENDIX TO APPENDIX D.

SECTION E.1

CALCULATION OF VARIOUS SERIES USED IN APPENDIX D

Section E.1.1 To find S1(N) which is

$$\frac{1}{N+1}-\frac{1+3}{N+3}+\frac{3+5}{N+5}-\frac{5+7}{N+7}+\cdots$$

This series does not converge, but finding the middle of the oscillation appears to give useful results. This is also true for $S_2(N)$, $S_1'(N)$, and $S_2'(N)$.

$$\begin{aligned}
S_1(N) &= \left(\frac{1}{N+1} - \frac{3}{N+3} + \frac{5}{N+5} - \frac{7}{N+7} + \ldots\right) \\
&\quad - \left(\frac{1}{n+3} - \frac{3}{N+5} + \frac{5}{N+7} - \frac{7}{N+9} + \ldots\right) \\
&= \left((1 - \frac{N}{N+1}) - (1 - \frac{N}{N+3}) + (1 - \frac{N}{N+5}) - \ldots\right) \\
&\quad - \left((1 - \frac{N+2}{N+3}) - (1 - \frac{N+2}{N+5}) = (1 - \frac{N+2}{N+7}) - \ldots\right) \\
&= \left((1-1+1-\ldots) - N(\frac{1}{N+1} - \frac{1}{N+3} + \frac{1}{N+5} - \ldots)\right) \\
&\quad - \left((1-1+1-\ldots) - (N+2)(\frac{1}{N+3} - \frac{1}{N+5} + \frac{1}{N+7} - \ldots)\right)
\end{aligned}$$

$$\underline{S_1(N) = -N\left(\frac{1}{N+1} - \frac{1}{N+3} + \frac{1}{N+5} - \ldots\right) + (N+2)\left(\frac{1}{N+3} - \frac{1}{N+5} + \ldots\right)}$$

Case 1 N=4I where I integer

$$\begin{aligned}
S_1(N) &= -N\left[\frac{\pi}{4} - \left(1-\frac{1}{3}+\frac{1}{5}-\ldots-\frac{1}{N-1}\right)\right] + (N+2)\left(\left(1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{1}{N+1}\right)-\frac{\pi}{4}\right) \\
&= -N\frac{\pi}{4} - (N+2)\frac{\pi}{4} + N\left(1-\frac{1}{3}+\frac{1}{5}\ldots-\frac{1}{N-1}\right) + (N+2)\left(1-\frac{1}{3}+\frac{1}{5}\ldots+\frac{1}{N+1}\right)
\end{aligned}$$

$$\underline{S_1(N) = 2(N+1)\left(-\frac{\pi}{4}+\left(1-\frac{1}{3}+\frac{1}{5}\ldots-\frac{1}{N-1}\right)\right) + \frac{N+2}{N+1}}$$

Case 2 N=4I+2

$$\underline{S_1(N) = 2(N+1)\left(-\frac{\pi}{4}+\left(1-\frac{1}{3}+\ldots+\frac{1}{N-1}\right)\right) + \frac{N+2}{N+1}}$$

Case 3 N=4I+1

$$S_1(N) = -N\left(\frac{1}{2}\log_e 2 - \left(\frac{1}{2}-\frac{1}{4}+\ldots+\frac{1}{N-1}\right)\right) + (N+2)\left(\left(\frac{1}{2}-\frac{1}{4}\ldots+\frac{1}{N+1}\right)-\frac{1}{2}\log_e 2\right)$$

$$\underline{S_1(N) = 2(N+1)\left(\left(\frac{1}{2}-\frac{1}{4}+\ldots-\frac{1}{N-1}\right)-\frac{1}{2}\log_e 2\right) + \frac{N+2}{N+1}}$$

Case 4 $N=4I+3$

$$S_1(N) = 2(N+1)\left(\frac{1}{2}\log_e 2-\left(\frac{1}{2}-\frac{1}{4}\cdots+\frac{1}{N-1}\right)\right) + \frac{N+2}{N+1}$$

Section E.1.2 To find $S_2(N)$ which is $\frac{2}{N+2} - \frac{2+4}{N+4} + \frac{4+6}{N+6} - \cdots$

similarly to $S_1(N)$

$$S_2(N) = -N\left(\frac{1}{N+2}-\frac{1}{N+4}+\frac{1}{N+6}\cdots\right) + (N+2)\left(\frac{1}{N+4}-\frac{1}{N+6}+\frac{1}{N+8}\cdots\right)$$

Case 1 $N=4I$

$$S_2(N) = 2(N+1)\left(-\frac{1}{2}\log_e 2-\left(\frac{1}{2}-\frac{1}{4}\cdots-\frac{1}{N}\right)\right) +1$$

Case 2 $N=4I+2$

$$S_2(N) = 2(N+1)\left(\frac{1}{2}\log_e 2-\left(\frac{1}{2}-\frac{1}{4}\cdots+\frac{1}{N}\right)\right) + 1$$

Case 3 $N=4I+1$

$$S_2(N) = 2(N+1)\left(\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}\cdots+\frac{1}{N}\right)\right) + 1$$

Case 4 $N=4I+3$

$$S_2(N) = 2(N+1)\left(\left(1-\frac{1}{3}+\frac{1}{5}\cdots-\frac{1}{N}\right)-\frac{\pi}{4}\right) + 1$$

Section E.1.3 To find $S_1'(N)$ and $S_2'(N)$

$$\frac{1}{N+1}-\frac{3}{N+3}+\frac{5}{N+5}-\cdots \qquad \frac{2}{N+2}-\frac{4}{N+4}+\frac{6}{N+6}-\frac{8}{N+8}+\cdots$$

$$S_1'(N) = \frac{1}{2} - N\left(\frac{1}{N+1}-\frac{1}{N+3}+\frac{1}{N+5}-\cdots\right)$$

$$S_2'(N) = \frac{1}{2} - N\left(\frac{1}{N+2}-\frac{1}{N\pm 4}+\frac{1}{N+6}-\cdots\right)$$

Case 1 $N=4I$

$$S_1'(N) = \frac{1}{2}-N\left(\frac{\pi}{4}-\left(1-\frac{1}{3}+\cdots-\frac{1}{N-1}\right)\right)$$

$$S_2'(N) = \frac{1}{2}-N\left(\frac{1}{2}\log_e 2-\left(\frac{1}{2}-\frac{1}{4}\ldots-\frac{1}{N}\right)\right)$$

Case 2 $N=4I+2$

$$S_1'(N) = \frac{1}{2}-N\left(\left(1-\frac{1}{3}\ldots+\frac{1}{N-1}\right)-\frac{\pi}{4}\right)$$
$$S_2'(N) = \frac{1}{2}-N\left(\left(\frac{1}{2}-\frac{1}{4}\ldots+\frac{1}{N}\right)-\frac{1}{2}\log_e 2\right)$$

Case 3 $N=4I+1$

$$S_1'(N) = \frac{1}{2}-N\left(\frac{1}{2}\log_e 2-\left(\frac{1}{2}-\frac{1}{4}\ldots+\frac{1}{N-1}\right)\right) \qquad N \geq 5$$
$$S_2'(N) = \frac{1}{2}-N\left(\left(1-\frac{1}{3}+\frac{1}{5}\ldots+\frac{1}{N}\right)-\frac{\pi}{4}\right)$$
$$S_1'(1) = \frac{1}{2} - \frac{1}{2}\log_e 2$$

Case 4 $N=4I+3$

$$S_1'(N) = \frac{1}{2}-N\left(\left(\frac{1}{2}-\frac{1}{4}\ldots+\frac{1}{N-1}\right)-\frac{1}{2}\log_e 2\right)$$
$$S_2'(N) = \frac{1}{2}-N\left(\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}\ldots-\frac{1}{N}\right)\right)$$

Section E.1.4 Calculation of $A = 1-\lambda_2+\lambda_4-\lambda_6+\lambda_8-\ldots$

$B' = \lambda_1-\lambda_3+\lambda_5-\lambda_7+\ldots$

Using $\left(\frac{1-\xi}{1+\alpha\xi}\right) = 1+\lambda_1\xi+\lambda_2\xi^2+\lambda_3\xi^3+\ldots$

$\xi = i$ gives $\left(\frac{1-i}{1+\alpha i}\right)^{1/2} = 1+i\lambda_1-\lambda_2-i\lambda_3+\lambda_4+i\lambda_5-\lambda_6+\ldots$

$\xi = -i$ gives $\left(\frac{1+i}{1-\alpha i}\right)^{1/2} = 1-i\lambda_1-\lambda_2+i\lambda_3+\lambda_4-i\lambda_5-\lambda_6+\ldots$

$$R\left(\frac{1-i}{1+\alpha i}\right)^{1/2} = R(1-\lambda_2+\lambda_4-\ldots) - Im(\lambda_1-\lambda_3+\lambda_5-\ldots) \quad \ldots(1)$$
$$Im\left(\frac{1-i}{1+\alpha i}\right)^{1/2} = Im(1-\lambda_2+\lambda_4\ldots) + R(\lambda_1-\lambda_3+\lambda_5-\ldots) \quad \ldots(2)$$
$$R\left(\frac{1+i}{1-\alpha i}\right)^{1/2} = R(1-\lambda_2+\lambda_4-\ldots) + Im(\lambda_1-\lambda_3+\lambda_5-\ldots) \quad \ldots(3)$$
$$Im\left(\frac{1+i}{1-\alpha i}\right)^{1/2} = Im(1-\lambda_2+\lambda_4-\ldots) - R(\lambda_1-\lambda_3+\lambda_5-\ldots) \quad \ldots(4)$$
$$\frac{(1)+(3)}{2} \rightarrow \frac{1}{2}\left\{R\left(\frac{1-i}{1+\ i}\right)^{1/2} + R\left(\frac{1+i}{1-\ i}\right)^{1/2}\right\} = R(1-\lambda_2+\lambda_4-\ldots)$$
$$\frac{(2)+(4)}{2} \rightarrow \frac{1}{2}\left\{Im\left(\frac{1-i}{1+\ i}\right)^{1/2} + Im\left(\frac{1+i}{1-\ i}\right)^{1/2}\right\} = Im(1-\lambda_2+\lambda_4-\ldots)$$

$\frac{(3)-(1)}{2} \rightarrow \frac{1}{2}\left\{R\left(\frac{1+i}{1-i\alpha}\right)^{1/2} - R\left(\frac{1-i}{1+\alpha i}\right)^{1/2}\right\} = Im(\lambda_1 - \lambda_3 + \lambda_5 - \ldots)$

$\frac{(4)-(2)}{2} \rightarrow \frac{1}{2}\left\{Im\left(\frac{1+i}{1-\alpha i}\right)^{1/2} - Im\left(\frac{1-i}{1+\alpha i}\right)^{1/2}\right\} = -R(\lambda_1 - \lambda_3 + \lambda_5 - \ldots)$

To find $\left(\frac{1-i}{1+\alpha i}\right)^{1/2}$, let $\alpha = U + iV$

then $\left(\frac{1-i}{1+\alpha i}\right)^{1/2} = \left(\frac{(1-V-U)}{2} - \frac{i(1-V+U)}{2V}\right)^{1/2}$

$= 1.646 + .456i$ when $U = -.4706$ $V = .8824$

Similarly $\left(\frac{1+i}{1-\alpha i}\right)^{1/2} = .8828 + .2280i$

So $R(1-\lambda_2+\lambda_4-\lambda_6+\ldots) = 1.23414$

$Im(1-\lambda_2+\lambda_4-\lambda_6+\ldots) = .34184$

$R(\lambda_1-\lambda_3+\lambda_5-\ldots) = .11395$

$Im(\lambda_1-\lambda_3+\lambda_5-\ldots) = -.41138$

from above theory using the programmes COSI to calculate the values

But $1-\lambda_2+\lambda_4-\ldots = A$

and $\lambda_1-\lambda_3+\lambda_5-\ldots = B'$

So $A = 1.23414 + .34184i$

$B' = .11395 - .41138i$

Section E.1.5 To find $\lambda_2+\frac{\lambda_3}{2}+\frac{\lambda_4}{3}+\frac{\lambda_5}{4}+\ldots$

$\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2} = 1+\lambda_1\xi+\lambda_2\xi^2+\lambda_3\xi^3+\ldots$

$\int\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2}\frac{1}{\xi^2}d\xi = -\frac{1}{\xi}+\lambda_1\log\xi+\lambda_2\xi+\frac{\lambda_3\xi^2}{2}+\ldots$ (1)

$\int\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2}\frac{1}{\xi^2}d\xi\Big|_{\xi=1} = -1+\lambda_2+\frac{\lambda_3}{2}+\frac{\lambda_4}{3}+\frac{\lambda_5}{4}+\ldots$

Now $\int\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2}\frac{1}{\xi^2}d\xi = -\sqrt{\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}+\alpha\right)} + (1+\alpha)\cosh^{-1}\sqrt{\left(\frac{1}{\xi}+\alpha\right)/(1+\alpha)} + C$

... (2) C is const. of intgn.

So $\int\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2}\frac{1}{\xi^2}d\xi\Big|_{\xi=1} = C$

but also $= -1+\lambda_2+\frac{\lambda_3}{2}+\frac{\lambda_4}{3}+\frac{\lambda_5}{4}+\ldots$

i.e. $\lambda_2 + \frac{\lambda_3}{2} + \frac{\lambda_4}{3} + \ldots = 1 + C$

Now find C.

By substituting a small value for ξ (say .1) in eqn. (1), we can obtain a very accurate value for the right hand side by taking only a few terms. Equating this to the left hand side then gives a numerical value for $\int\left(\frac{1-\xi}{1+\alpha\xi}\right)^{1/2}\frac{1}{\xi^2}d\xi\Big|_{\xi=.1}$

This must then equal the r.h.s. of eqn. (2), obtained by substituting .1 in it.

Thus, C can be calculated.

The r.h.s. of eqn. (1), with $\xi = .1$,

$$= -9.43712 + 1.00808i \text{ (as calculated by COSI(2))}$$

The r.h.s. of eqn. (2), with $\xi = .1$,

$$= -7.88185 + .89315i + C$$

This gives $C = -1.54927 + .11493i$

so $\lambda_2 + \frac{\lambda_3}{2} + \frac{\lambda_4}{3} + \ldots = -.54927 + .11493i$.

SECTION E.2

DESCRIPTION OF FORTRAN PROGRAMMES USED.

Section E.2.1 FORTRAN programme COSI(1) Case 2B

The programme first reads the value of $a = (k/h)^2 - 4$ in this case. It then calculates $\alpha = -\frac{1-ai}{a-i}$ and hence $\alpha^2, \alpha^3, \ldots, \alpha^{199}$.

Lines 1 to 16 of the printout show the real values of $\alpha^0, \alpha^1, \ldots, \alpha^{199}$ and lines 17 to 32 show the imaginary values of $\alpha^0, \alpha^1, \ldots, \alpha^{199}$.

It then calculates $\lambda_1, \lambda_2, \ldots, \lambda_{199}$ by the method on p.105

and also calculates

$1+\lambda_1$, $1+\lambda_1+\lambda_2$, $1+\lambda_1+\lambda_2+\lambda_3$, ...

The real values of λ_1 to λ_{199} are then printed (lines 33 to 48), then the imaginary values (lines 49 to 64) and then the real and imaginary values of $\sum_{r=0}^{N}\lambda_r$ for N = 1 t0 199 (lines 65 to 80, and 81 to 96).

Note that $\sum_{r=0}^{N}\lambda_r$ must be close to 0 since the infinite sum is 0. However λ_N where N is as great as the programme makes it is still far from insignificant and so the finite sum is large enough to be significant.

Next the programme calculates B where

$$B = \frac{2Ai(1-i)^{1/2}}{(a-i)^{1/2}}$$ see p. 68

The value is .62533 + 2.25764i and is precisely that calculated by CALC, which did not take advantage of the FORTRAN compiler's ability to handle complex variables.

Next the programme calculates $S_1(1)$, $S_1(2)$,..., $S_1(53)$ and $S_2(1)$, $S_2(2)$,..., $S_2(53)$ and prints these (lines 98 to 102 and 103 to 107). It then calculates but does not print

$S_1'(1)$, $S_1'(2)$, $S_1'(3)$, $S_1'(4)$ and

$S_2'(1)$, $S_2'(2)$, $S_2'(3)$, $S_2'(4)$.

It then calculates and prints A and B' as shown on pages 136 and 137. Then are shown the real values of ϕ_1, ϕ_2 ,..., ϕ_{52} and the imaginary values of same as calculated on p.110 , then the real and imaginary values of ϕ_1', ϕ_2', ϕ_3', ϕ_4' as shown on p. (lines 218 and 219)

Then are shown the real values of $\frac{C_N + C_{N+2}}{-B}$ N=1,52 and the corresponding imaginary values. These are derived from eqns. * on p. 111.

Next are shown A_1, A_2, A_3,..., A_{104} (see p.73), then the real and imaginary parts of C_1, C_2, C_3, C_4 as derived from the eqns. on pp. 112 - 113. Then are calculated Z(1) to Z(108) (lines 238 to 246), which are the amounts which must be added to the r.h.s.'s of the adjusted equations (assuming initial values of b_n) to give the correct r.h.s.'s.

From the A_1, A_2, etc. then are calculated the coeffs. of the unadjusted eqns. and hence the coeffs. of the adjusted eqns.. viz.

eqn. 1	=	sixth eqn.	-	first eqn.	
eqn. 3	=	first eqn.	-	eighth eqn.	
eqn. 4	=	third eqn.	+	fourth eqn.	
eqn. 5	==	second eqn.	+	sixth eqn.	
eqn. 6	=	first eqn.	+	fifth eqn.	etc.

(These coefficients are, of course, the same as for CALC - Case2)

The amounts which have to be added to the initial values of the x's are then found in the normal way.

Section E.2.2 FORTRAN programme COSI(2) Case 2A

The programme calculates the α_n and λ_n as does COSI(1). It then calculates $1+\lambda_1$, $1+\lambda_1+\lambda_2$, ...

i.e. $\sum_{r=0}^{N} \lambda_r$ for N= 1 to 60 only and then PD(1),PD(2),...PD(60)

where $PD(1) = -1+\lambda_2+\lambda_3/2+\lambda_4/3+...+\lambda_{130}/129$

$PD(2) = -1+\lambda_2+\lambda_3/2+ \quad ...+\lambda_{130}/128+\lambda_{131}/130$

$PD(60) = -1+\lambda_2+ \quad ... \quad +\lambda_{189}/188$

All these values are, of course, very close to the constant C calculated on p. . The real parts and the imaginary parts of PD(1) to PD(60) are shown on the listing.

Then are calculated the realparts and the imaginary parts of $\lambda_1, 1-\lambda_2, \lambda_1-\lambda_3, 1-\lambda_2+\lambda_4, ..., 1-\lambda_2+\lambda_4-...+\lambda_{100}$

The higher terms of these are, of course, approximately equal to A and B' respectively, as calculated on p.136 .

It then prints $S'(1)$ to $S'(52)$ and then $T_1(1)$ to $T_1(52)$.

These are extremely small, as stated on p. 122.

It then prints the r.h.s. of eqn. (1) with $\xi = .1$

The next line is the real and imaginary parts of $\sqrt{\frac{\frac{1}{\xi}+\alpha}{1+\alpha}}$

when $\xi = .1$ VIZ 2.72034 - 1.37850i

The next but one line is

$$\cosh^{-1} \sqrt{\frac{\frac{1}{\xi}+\alpha}{1+\alpha}} \quad \text{VIZ} \quad 1.79268 - 0.49175$$

and due to possible ambiguity of signs taken in the function $\left(\cosh^{-1}Z = \log\,(Z \pm i\sqrt{1-Z^2}\,)\right)$, the correctness of this function is checked by calculating and printing the cosh of this value VIZ 2.72034 - 1.37850i, which is correct.

C is then printed and followed by A and B' as for COSI(1). This is followed by the real values of $\phi(1)$ to $\phi(52)$ and the imaginary values of $\phi(1)$ to $\phi(52)$.

Then is printed the real values of $\phi'(1)$, $\phi'(2)$, $\phi'(3)$ and $\phi'(4)$ and then the corresponding imaginary values.

From this the real values of $\frac{C_N+C_{N+2}}{-B}$ N = 1 to 52 and the corresponding imaginary values are shown as for COSI(1).

These are quite close to the COSI(1) values.

The rest of COSI(2) is the same as COSI(1).

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